

APPROXIMATE CONTROLLABILITY OF SECOND ORDER IMPULSIVE FUNCTIONAL DIFFERENTIAL SYSTEM WITH INFINITE DELAY IN BANACH SPACES*

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Abstract This paper studies the approximate controllability of second order impulsive functional differential system with infinite delay in Banach spaces. Sufficient conditions are formulated and proved for the approximate controllability of such system under the assumption that the associated linear part of system is approximately controllable. The results are obtained by using strongly continuous cosine families of operators and the contraction mapping principle. An example is given to illustrate the obtained theory.

Keywords Approximate controllability, strongly continuous cosine families of operators, impulsive functional differential system, infinite delay

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1. Introduction

Second order abstract differential systems arise in many fields such as mathematical physics and engineering, and have been extensively studied during the past few decades. The problem of controllability for second order differential systems in Banach spaces has received considerable attention recently. Kang et al. [16] studied the exact controllability for the second order differential inclusion in Banach spaces. With the help of a fixed point theorem for condensing maps due to Martelli, the authors found a control $u(\cdot)$ in $L^2(J, U)$ such that the solution satisfies $x(b) = x_1$ and $x'(b) = y_1$. Their results depend on the following two conditions:

- (a) The associated sine family $\{S(t), t \in R\}$ of operators is compact.
- (b) The linear operators

$$G_1 u = \int_0^T S(T-s)Bu(s)ds$$

and

$$G_2 u = \int_0^T C(T-s)Bu(s)ds$$

are invertible and their inverses are bounded.

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Balachandran and Kim [3] corrected an error of the control function in [16] and pointed out that the compactness assumption of the sine family $S(t)$ and the conditions on G_1 imply that the operator G_1 is compact and surjective, and thus by the application of Baire Category theorem X is finite dimensional [26].

Chang and Li [5] investigated the exact controllability problem for a class of second order differential and integro-differential inclusions in Banach space. They defined a control function which was similar to that in [16] and established exact controllability results. Their results also require G_1 to be invertible and its inverse to be bounded without imposing a compactness condition on the sine family $S(t)$. However, in most of real control systems the operators $S(t)$ are compact for $t \in R$ [14, Theorem 3.2], thus their assumptions restricted the state space to finite dimensional, and the examples recovered from the abstract theory only pertain to ordinary differential equations. Recently, Henríquez and Cuevas [12] studied the approximate controllability of control systems with state and control in Banach spaces and described by a second order semilinear abstract differential equation. They compared the approximate controllability of the system with the approximate controllability of an associated discrete system. The main assumption in [12] is the approximate controllability of the corresponding linear system. To illustrate the proposed result, they applied the theory to a wave equation.

In real systems, signal processing may introduce delays. The approximate controllability results for second order semilinear abstract functional differential equations with infinite delay was shown in [13] under the assumption that the corresponding linear system is approximate controllable. In recent years, the study of impulsive second order control systems has received increasing interest, since dynamical systems with impulsive effects have numerous applications to problems arising in information sciences, electronics, biology, ecology, etc. Sakthivel et al. [21] studied the exact controllability of second order nonlinear impulsive differential systems by using a fixed point analysis approach. Moreover, Sakthivel et al. [22] obtained the approximate controllability results for second order stochastic differential equations with impulsive effects under the assumption that the associated linear system is approximately controllable. Unfortunately, in these two papers, the authors didn't consider the damped term $x'(\cdot)$ in defining the exact and approximate controllability of the corresponding systems, which violate the controllability definition "the state variable steers some initial position to final one" because $x'(t)$ is a state variable for a second order system. Motivated by [3], Radhakrishnan and Balachandran [19] discussed the exact controllability of second order neutral integro-differential equations with impulsive conditions in Banach spaces. For the same reason as described above for [5], the results in those papers are only applicable to ordinary differential equations.

More recently, Arthi and Balachandran [2] investigated the exact controllability of second order impulsive evolution systems with infinite delay. However, they only considered $x(t)$ without taking into account the damped term $x'(t)$ in defining the exact controllability of the second order abstract system. Up to now, to the authors' knowledge, controllability of such systems with proper definition has not been studied.

The concept of exact controllability is usually too strong and has limited applicability. Approximate controllable systems are more prevalent, and very often the results are adequate in application. Therefore, it is necessary and important to consider approximate controllability for second order impulsive functional differ-

ential systems with infinite delay. In this paper, we derive results on approximate controllability of second order impulsive functional differential system with infinite delay by assuming that nonlinear function and impulses satisfy some inequality conditions, and the corresponding linear system is approximate controllable. The system considered in this paper is a generalization of those without delay or impulses that were studied in [2, 12, 13, 19, 21, 22]. More precisely, we consider the following semilinear system:

$$\begin{aligned} x''(t) &= Ax(t) + Bu(t) + f(t, x_t, x'_t), \quad t \in J, \quad t \neq t_k, \\ \Delta x|_{t=t_k} &= I_k^1(x(t_k)), \quad k = 1, \dots, m, \\ \Delta x'|_{t=t_k} &= I_k^2(x'(t_k^+)), \quad k = 1, \dots, m, \\ x_0 &= \phi \in \mathcal{B}, \quad x'_0 = \varphi \in \mathcal{B}, \end{aligned} \tag{1.1}$$

where $J = [0, b]$, the state $x(\cdot)$ takes values in a Banach space X with the norm $\|\cdot\|$, $u(\cdot) \in L^2(J, U)$ is the control function where U is a Banach space, A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ on X , $B : U \rightarrow X$ is a bounded linear operator. $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = b$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$, and $I_k^j : X \rightarrow X$, $j = 1, 2$, $f : J \times \mathcal{B} \times \mathcal{B} \rightarrow X$ are appropriate continuous functions to be specified later. The histories $x_t, x'_t : (-\infty, 0] \rightarrow X$, $x_t(\theta) = x(t + \theta)$, $x'_t(\theta) = x'(t + \theta)$, $\theta \leq 0$, belong to some abstract phase space \mathcal{B} defined axiomatically.

The paper is organized as follows. In Section 2, we recall some fundamental concepts and establish existence of mild solutions for system (1.1). In section 3, we present some criteria for the approximate controllability of system (1.1) in terms of the system defined by the linear part. Finally, in Section 4, an example is presented which illustrates the main theorem.

2. Preliminaries

In this section, we review some basic concepts, notations and properties needed to establish our results.

Definition 2.1. (see [23, 24]) A one parameter family $\{C(t) : t \in R\}$, of bounded linear operators in the Banach space X is called a strongly continuous cosine family iff

- (i) $C(s+t) + C(s-t) = 2C(s)C(t)$ for all $s, t \in R$;
- (ii) $C(0) = I$;
- (iii) $C(t)x$ is strongly continuous in t on R for each fixed $x \in X$.

Throughout this paper, A is the infinitesimal generator of a strongly continuous cosine family, $\{C(t) : t \in R\}$, of bounded linear operators defined on a Banach space X endowed with a norm $\|\cdot\|$. We denote by $\{S(t) : t \in R\}$ the sine function associated to $\{C(t) : t \in R\}$ which is defined by

$$S(t)x = \int_0^t C(s)x ds, \quad x \in X, \quad t \in R.$$

Moreover, M and N are positive constants such that $\|C(t)\| \leq M$ and $\|S(t)\| \leq N$ for every $t \in J$.

The infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ is the operator $A : X \rightarrow X$ defined by

$$Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}, \quad x \in D(A),$$

where $D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable in } t\}$, endowed with the norm

$$\|x\|_A = \|x\| + \|Ax\|, \quad x \in D(A).$$

Define $E = \{x \in X : C(t)x \text{ is once continuously differentiable in } t\}$, endowed with the norm

$$\|x\|_E = \|x\| + \sup_{0 \leq t \leq 1} \|AS(t)x\|, \quad x \in E,$$

then E is a Banach space. The operator-valued function

$$\mathcal{H}(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$$

is a strongly continuous group of bounded linear operators on the space $E \times X$

generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. From this, it

follows that $AS(t) : E \rightarrow X$ is a bounded linear operator and that $AS(t)x \rightarrow 0$ as $t \rightarrow 0$ for each $x \in E$. Furthermore, [25]

$$S(t + s) = C(t)S(s) + C(s)S(t), \tag{2.1}$$

$$C(t + s) = C(t)C(s) + AS(s)S(t), \tag{2.2}$$

$$AS(s)S(t) = \frac{1}{2}[C(t + s) - C(t - s)]. \tag{2.3}$$

The following properties are well known [23]:

- (i) if $x \in X$ then $S(t)x \in E$ for every $t \in R$;
- (ii) if $x \in E$ then $S(t)x \in D(A)$, $\frac{d}{dt}C(t)x = AS(t)x$ and $\frac{d^2}{dt^2}S(t)x = AS(t)x$ for every $t \in R$;
- (iii) if $x \in D(A)$ then $C(t)x \in D(A)$ and $\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax$ for every $t \in R$;
- (iv) if $x \in D(A)$ then $S(t)x \in D(A)$ and $\frac{d^2}{dt^2}S(t)x = AS(t)x = S(t)Ax$ for every $t \in R$.

The existence of solutions of the second order abstract Cauchy problem,

$$\begin{aligned} x''(t) &= Ax(t) + h(t), \quad t \in J, \\ x(0) &= \varsigma_0, \\ x'(0) &= \varsigma_1, \end{aligned} \tag{2.4}$$

where $h : J \rightarrow X$ is an integral function, has been discussed in [23]. Similarly, the existence of solutions of semilinear second order abstract cauchy problems has been treated in [24]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)\zeta_0 + S(t)\zeta_1 + \int_0^t S(t-s)h(s)ds, \quad t \in J, \quad (2.5)$$

is called a mild solution of (2.4), and that when $\zeta_1 \in E$ the function $x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t)\zeta_0 + C(t)\zeta_1 + \int_0^t C(t-s)h(s)ds, \quad t \in J. \quad (2.6)$$

A function $u : [\sigma, \tau] \rightarrow X$ is said to be a normalized piecewise continuous function on $[\sigma, \tau]$ if u is piecewise continuous and left continuous on $(\sigma, \tau]$. We denote by $\mathcal{PC}([\sigma, \tau], X)$ the space of normalized piecewise continuous functions from $[\sigma, \tau]$ into X . In particular, we introduce the space \mathcal{PC} formed by all normalized piecewise continuous functions $u : J \rightarrow X$ such that u is continuous at $t \neq t_k, k = 1, \dots, m$. It is clear that \mathcal{PC} endowed with the norm $\|u\|_{\mathcal{PC}} = \sup_{t \in J} \|u(t)\|$ is a Banach space, where $\|\cdot\|$ is any norm of X .

Throughout, we set $t_0 = 0, t_{m+1} = b$, and for $u \in \mathcal{PC}$ we denote by \tilde{u}_k , for $k = 0, 1, \dots, m$, the function $\tilde{u}_k \in C([t_k, t_{k+1}]; X)$ given by $\tilde{u}_k(t) = u(t)$ for $t \in (t_k, t_{k+1}]$ and $\tilde{u}_k(t_k) = \lim_{t \rightarrow t_k^+} u(t)$.

A normalized piecewise continuous function $x : [\sigma, \tau] \rightarrow X$ is said to be normalized piecewise smooth on $[\sigma, \tau]$ if x is continuously differentiable except on a finite set S , the left derivative exists on $(\sigma, \tau]$ and the right derivative exists on $[\sigma, \tau)$. In this case, we present by $x'(t)$ the left derivative at $t \in (\sigma, \tau]$ and by $x'(\sigma)$ the right derivative at σ . We denote by $\mathcal{PC}^1([\sigma, \tau], X)$ the space of normalized piecewise smooth functions from $[\sigma, \tau]$ into X and by \mathcal{PC}^1 the space formed by all normalized piecewise smooth functions $x : J \rightarrow X$ such that $S = \{t_k : k = 1, \dots, m\}$. Obviously, \mathcal{PC}^1 is also a Banach space with the norm $\|x\|_{\mathcal{PC}^1} = \max\{\|x\|_{\mathcal{PC}}, \|x'\|_{\mathcal{PC}}\}$.

In this paper we employ an axiomatic definition of the phase space \mathcal{B} which is similar to that introduced by Hale and Kato [10] and appropriated to treat retarded impulsive differential equations. Specifically, \mathcal{B} is a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$. We assume that \mathcal{B} satisfies the following axioms:

(A) If $x : (-\infty, \sigma + b] \rightarrow X$, $b > 0$, is such that $x_\sigma \in \mathcal{B}$ and $x|_{[\sigma, \sigma+b]} \in \mathcal{PC}([\sigma, \sigma + b], X)$, then for every $t \in [\sigma, \sigma + b)$ the following conditions hold:

- (i) x_t is in \mathcal{B} ;
- (ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$;
- (iii) $\|x_t\|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t - \sigma)\|x_\sigma\|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \rightarrow [1, \infty)$, K is continuous, M is locally bounded, and H, K, M are independent of $x(\cdot)$.

(B) The space \mathcal{B} is complete.

Example 2.1. (The phase space $\mathcal{PC}_r \times L^p(\rho, X)$). Let $r \geq 0, 1 \leq p < \infty$ and let $\rho : (-\infty, -r] \rightarrow R$ be a non-negative measurable function. Assume that ρ satisfy the following conditions:

- (i) For every $\xi \in (-\infty, -r)$, $\int_{\xi}^{-r} \rho(\theta)d\theta < \infty$,
- (ii) There exists a non-negative, locally bounded function ζ on $(-\infty, 0]$ such that $\rho(\xi+\theta) \leq \zeta(\xi)\rho(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_{\xi}$, where $N_{\xi} \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero.

The space $\mathcal{B} = \mathcal{PC}_r \times L^p(\rho, X)$ consists of all classes of Lebesgue-measurable functions $\psi : (-\infty, 0] \rightarrow X$ such that $\psi|_{[-r, 0]} \in \mathcal{PC}([-r, 0], X)$ and $\rho\|\psi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $\mathcal{PC}_r \times L^p(\rho, X)$ is defined by

$$\|\psi\|_{\mathcal{B}} = \sup\{\|\psi(\theta)\| : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} \rho(\theta)\|\psi(\theta)\|^p d\theta\right)^{1/p}.$$

Proceeding as in the proof of [15, Theorem 1.3.8], it follows that \mathcal{B} is a space which satisfies the axioms (A) and (B). Moreover, when $r = 0$ and $p = 2$, $\mathcal{B} = \mathcal{PC}_0 \times L^2(\rho, X)$ coincides with $C_0 \times L^p(\rho, X)$. We can take $H = 1$, $M(t) = \zeta(-t)^{1/2}$ and $K(t) = 1 + \left(\int_{-t}^0 \rho(\theta)d\theta\right)^{1/2}$ for $t \geq 0$.

In order to define the solution of the system (1.1), we consider the space

$$\begin{aligned} \mathcal{B}'_{h_1} &= \{x : (-\infty, b] \rightarrow X \text{ such that } x(\cdot)|_J \in \mathcal{PC}, x_0 \in \mathcal{B}\} \text{ and} \\ \mathcal{B}'_{h_2} &= \{x : (-\infty, b] \rightarrow X \text{ such that } x \text{ and } x' \in \mathcal{B}'_{h_1}\}. \end{aligned}$$

Set $\|\cdot\|_{\mathcal{B}'_{h_1}}, \|\cdot\|_{\mathcal{B}'_{h_2}}$ to be the seminorm in \mathcal{B}'_{h_1} and \mathcal{B}'_{h_2} defined by

$$\|x\|_{\mathcal{B}'_{h_1}} = \|\phi\|_{\mathcal{B}} + \sup_{s \in J} \|x(s)\|, \quad x \in \mathcal{B}'_{h_1}$$

and

$$\|x\|_{\mathcal{B}'_{h_2}} = \max\{\|x\|_{\mathcal{B}'_{h_1}}, \|x'\|_{\mathcal{B}'_{h_1}}\}, \quad x \in \mathcal{B}'_{h_2}.$$

Motivated by the formula (2.5), we give the mild solution for the problem (1.1).

Definition 2.2. A function $x(\cdot, \phi, \varphi, u) \in \mathcal{B}'_{h_2}$ is said to be a mild solution of (1.1) if

- (i) $x_0 = \phi \in \mathcal{B}, x'_0 = \varphi \in \mathcal{B}$;
- (ii) $\Delta x|_{t=t_k} = I_k^1(x(t_k)), k = 1, \dots, m$;
- (iii) $\Delta x'|_{t=t_k} = I_k^2(x'(t_k^+)), k = 1, \dots, m$;
- (iv) $x(\cdot)|_J \in \mathcal{PC}^1$ and the following integral equation is verified:

$$\begin{aligned} x(t) &= C(t)\phi(0) + S(t)\varphi(0) + \int_0^t S(t-s)[Bu(s) + f(s, x_s, x'_s)]ds \\ &+ \sum_{0 < t_k < t} C(t-t_k)I_k^1(x(t_k)) + \sum_{0 < t_k < t} S(t-t_k)I_k^2(x'(t_k^+)), t \in J. \end{aligned} \tag{2.7}$$

To establish our results, we introduce the following assumptions on system (1.1):

(H₁) For each $0 \leq t < b$, the operator $\alpha(\alpha I + \Gamma_t^b)^{-1} \rightarrow 0$ in the strong operator topology as $\alpha \rightarrow 0^+$, where

$$\Gamma_t^b = \int_t^b S(b-s)BB^*S^*(b-s)ds$$

is the controllability Grammian.

(H₂) $f : J \times \mathcal{B} \times \mathcal{B} \rightarrow X$ is a continuous function and there exist positive constants k_1 and k_2 such that

$$\|f(t, \omega_1, \nu_1) - f(t, \omega_2, \nu_2)\| \leq k_1 \|\omega_1 - \omega_2\|_{\mathcal{B}} + k_2 \|\nu_1 - \nu_2\|_{\mathcal{B}}$$

for every $\omega_1, \omega_2, \nu_1$ and $\nu_2 \in \mathcal{B}$.

(H₃) The functions $I_k^j : X \rightarrow X$ are continuous and there exist positive constants $L(I_k^j)$, $j = 1, 2$ such that

$$\|I_k^j(x_1) - I_k^j(x_2)\| \leq L(I_k^j) \|x_1 - x_2\|$$

for each $x_1, x_2 \in X$.

(H₄) $\max\{\phi_1, \phi_2\} < 1$, where

$$\begin{aligned} \phi_1 &= \beta [Nb(k_1 + k_2)K_b + M \sum_{k=1}^m L(I_k^1) + N \sum_{k=1}^m L(I_k^2)], \\ \phi_2 &= Mb(k_1 + k_2)K_b + \eta [Nb(k_1 + k_2)K_b + M \sum_{k=1}^m L(I_k^1) + N \sum_{k=1}^m L(I_k^2)] \\ &\quad + \tilde{N} \sum_{k=1}^m L(I_k^1) + M \sum_{k=1}^m L(I_k^2), \end{aligned}$$

and $\beta = 1 + \frac{1}{\alpha} N^2 K^2 b$, $\eta = \frac{1}{\alpha} M N K^2 b$.

Theorem 2.1. *If the conditions (H₁)-(H₄) are satisfied, then the system (1.1) has a mild solution on J for all $u \in L^2(J, U)$.*

Proof. Let $l_f = \max_{t \in J} \|f(t, 0, 0)\|$ and $\|B\| \leq K$. Define the feedback control function

$$\begin{aligned} u(t) &= B^* S^*(b-t)(\alpha I + \Gamma_0^b)^{-1} [x_1 - C(b)\phi(0) - S(b)\varphi(0) - \int_0^b S(b-s)f(s, x_s, x'_s) ds \\ &\quad - \sum_{k=1}^m C(b-t_k)I_k^1(x(t_k)) - \sum_{k=1}^m S(b-t_k)I_k^2(x'(t_k^+))]. \end{aligned}$$

For $\phi \in \mathcal{B}$, we define $\tilde{\phi}$ by

$$\tilde{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ C(t)\phi(0) + S(t)\varphi(0), & t \in J, \end{cases}$$

and then $\tilde{\phi} \in \mathcal{B}'_{h_1}$.

For $\varphi \in \mathcal{B}$, we define $\tilde{\varphi}$ by

$$\tilde{\varphi}(t) = \begin{cases} \varphi(t), & t \in (-\infty, 0], \\ AS(t)\phi(0) + C(t)\varphi(0), & t \in J, \end{cases}$$

and then $\tilde{\varphi} \in \mathcal{B}'_{h_1}$.

Let $x(t) = \tilde{x}(t) + \tilde{\phi}(t)$, $x'(t) = \tilde{x}'(t) + \tilde{\varphi}(t)$, $-\infty < t \leq b$. It is easy to see that x satisfies

$$\begin{aligned} x(t) = & C(t)\phi(0) + S(t)\varphi(0) + \int_0^t S(t-s)f(s, x_s, x'_s)ds \\ & + \int_0^t S(t-\eta)BB^*S^*(b-\eta)(\alpha I + \Gamma_0^b)^{-1} \\ & \cdot [x_1 - C(b)\phi(0) - S(b)\varphi(0) - \int_0^b S(b-s)f(s, x_s, x'_s)ds \\ & - \sum_{k=1}^m C(b-t_k)I_k^1(x(t_k)) - \sum_{k=1}^m S(b-t_k)I_k^2(x'(t_k^+))]d\eta \\ & + \sum_{0 < t_k < t} C(t-t_k)I_k^1(x(t_k)) + \sum_{0 < t_k < t} S(t-t_k)I_k^2(x'(t_k^+)), \quad t \in J \end{aligned}$$

if and only if \tilde{x} satisfies $\tilde{x}_0 = 0$ and

$$\begin{aligned} \tilde{x}(t) = & \int_0^t S(t-s)f(s, \tilde{x}_s + \tilde{\phi}_s, \tilde{x}'_s + \tilde{\varphi}_s)ds \\ & + \int_0^t S(t-\eta)BB^*S^*(b-\eta)(\alpha I + \Gamma_0^b)^{-1} \\ & \cdot [x_1 - C(b)\phi(0) - S(b)\varphi(0) - \int_0^b S(b-s)f(s, \tilde{x}_s + \tilde{\phi}_s, \tilde{x}'_s + \tilde{\varphi}_s)ds \\ & - \sum_{k=1}^m C(b-t_k)I_k^1(\tilde{x}(t_k) + \tilde{\phi}(t_k)) - \sum_{k=1}^m S(b-t_k)I_k^2(\tilde{x}'(t_k^+) + \tilde{\varphi}(t_k^+))]d\eta \\ & + \sum_{0 < t_k < t} C(t-t_k)I_k^1(\tilde{x}(t_k) + \tilde{\phi}(t_k)) + \sum_{0 < t_k < t} S(t-t_k)I_k^2(\tilde{x}'(t_k^+) + \tilde{\varphi}(t_k^+)), \quad t \in J. \end{aligned}$$

It is also easy to see that x' satisfies

$$\begin{aligned} x'(t) = & AS(t)\phi(0) + C(t)\varphi(0) + \int_0^t C(t-s)f(s, x_s, x'_s)ds \\ & + \int_0^t C(t-\eta)BB^*S^*(b-\eta)(\alpha I + \Gamma_0^b)^{-1} \\ & \cdot [x_1 - C(b)\phi(0) - S(b)\varphi(0) - \int_0^b S(b-s)f(s, x_s, x'_s)ds \\ & - \sum_{k=1}^m C(b-t_k)I_k^1(x(t_k)) - \sum_{k=1}^m S(b-t_k)I_k^2(x'(t_k^+))]d\eta \\ & + \sum_{0 < t_k < t} AS(t-t_k)I_k^1(x(t_k)) + \sum_{0 < t_k < t} C(t-t_k)I_k^2(x'(t_k^+)), \quad t \in J \end{aligned}$$

if and only if \tilde{x}' satisfies $\tilde{x}'_0 = 0$ and

$$\begin{aligned} \tilde{x}'(t) = & \int_0^t C(t-s)f(s, \tilde{x}_s + \tilde{\phi}_s, \tilde{x}'_s + \tilde{\varphi}_s)ds \\ & + \int_0^t C(t-\eta)BB^*S^*(b-\eta)(\alpha I + \Gamma_0^b)^{-1} \end{aligned}$$

$$\begin{aligned}
& \cdot [x_1 - C(b)\phi(0) - S(b)\varphi(0) - \int_0^b S(b-s)f(s, \tilde{x}_s + \tilde{\phi}_s, \tilde{x}'_s + \tilde{\varphi}_s)ds \\
& - \sum_{k=1}^m C(b-t_k)I_k^1(\tilde{x}(t_k) + \tilde{\phi}(t_k)) - \sum_{k=1}^m S(b-t_k)I_k^2(\tilde{x}'(t_k^+) + \tilde{\varphi}(t_k^+))]d\eta \\
& + \sum_{0 < t_k < t} AS(t-t_k)I_k^1(\tilde{x}(t_k) + \tilde{\phi}(t_k)) + \sum_{0 < t_k < t} C(t-t_k)I_k^2(\tilde{x}'(t_k^+) + \tilde{\varphi}(t_k^+)), \quad t \in J.
\end{aligned}$$

Let $\mathcal{B}''_{h_1} = \{\tilde{x} \in \mathcal{B}'_{h_1} : \tilde{x}_0 = 0 \in \mathcal{B}\}$. For any $\tilde{x} \in \mathcal{B}''_{h_1}$,

$$\begin{aligned}
\|\tilde{x}\|_{\mathcal{B}''_{h_1}} &= \|\tilde{x}_0\|_{\mathcal{B}} + \sup_{s \in J} \|\tilde{x}(s)\| \\
&= \sup_{s \in J} \|\tilde{x}(s)\|,
\end{aligned}$$

and thus $(\mathcal{B}''_{h_1}, \|\cdot\|_{\mathcal{B}''_{h_1}})$ is a Banach space.

Let $Z = \mathcal{B}''_{h_1} \times \mathcal{B}''_{h_1}$ be the space

$$Z = \{(\tilde{x}, \tilde{z}) : \tilde{x}, \tilde{z} \in \mathcal{B}''_{h_1} \text{ and } \tilde{x}'(t) = \tilde{z}(t) \text{ for } t \neq t_k\}$$

provided with the norm

$$\|(\tilde{x}, \tilde{z})\|_Z = \max\{\|\tilde{x}\|_{\mathcal{B}''_{h_1}}, \|\tilde{z}\|_{\mathcal{B}''_{h_1}}\}.$$

It is now shown, $(\tilde{x}, \tilde{z}) \in Z$ implies $\tilde{x} \in \mathcal{B}'_{h_2}$.

On the space Z , we define the nonlinear operator

$$\Phi(\tilde{x}, \tilde{z}) = (\Phi_1(\tilde{x}, \tilde{z}), \Phi_2(\tilde{x}, \tilde{z})),$$

where

$$\begin{aligned}
\Phi_1(\tilde{x}, \tilde{z})(t) &= \int_0^t S(t-s)f(s, \tilde{x}_s + \tilde{\phi}_s, \tilde{x}'_s + \tilde{\varphi}_s)ds \\
& + \int_0^t S(t-\eta)BB^*S^*(b-\eta)(\alpha I + \Gamma_0^b)^{-1} \\
& \cdot [x_1 - C(b)\phi(0) - S(b)\varphi(0) - \int_0^b S(b-s)f(s, \tilde{x}_s + \tilde{\phi}_s, \tilde{x}'_s + \tilde{\varphi}_s)ds \\
& - \sum_{k=1}^m C(b-t_k)I_k^1(\tilde{x}(t_k) + \tilde{\phi}(t_k)) - \sum_{k=1}^m S(b-t_k)I_k^2(\tilde{x}'(t_k^+) + \tilde{\varphi}(t_k^+))]d\eta \\
& + \sum_{0 < t_k < t} C(t-t_k)I_k^1(\tilde{x}(t_k) + \tilde{\phi}(t_k)) + \sum_{0 < t_k < t} S(t-t_k)I_k^2(\tilde{x}'(t_k^+) + \tilde{\varphi}(t_k^+)),
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
 \Phi_2(\tilde{x}, \tilde{z})(t) &= \int_0^t C(t-s)f(s, \tilde{x}_s + \tilde{\phi}_s, \tilde{x}'_s + \tilde{\varphi}_s)ds \\
 &+ \int_0^t C(t-\eta)BB^*S^*(b-\eta)(\alpha I + \Gamma_0^b)^{-1} \\
 &\cdot [x_1 - C(b)\phi(0) - S(b)\varphi(0) - \int_0^b S(b-s)f(s, \tilde{x}_s + \tilde{\phi}_s, \tilde{x}'_s + \tilde{\varphi}_s)ds \\
 &- \sum_{k=1}^m C(b-t_k)I_k^1(\tilde{x}(t_k) + \tilde{\phi}(t_k)) - \sum_{k=1}^m S(b-t_k)I_k^2(\tilde{x}'(t_k^+) + \tilde{\varphi}(t_k^+))]d\eta \\
 &+ \sum_{0 < t_k < t} AS(t-t_k)I_k^1(\tilde{x}(t_k) + \tilde{\phi}(t_k)) + \sum_{0 < t_k < t} C(t-t_k)I_k^2(\tilde{x}'(t_k^+) + \tilde{\varphi}(t_k^+)).
 \end{aligned} \tag{2.9}$$

The continuity and well-definedness of Φ follow directly from the assumptions. It will be shown that the operator Φ has a fixed point.

Let $Q = \{(\tilde{x}, \tilde{z}) \in Z : \|(\tilde{x}, \tilde{z})\|_Z \leq r\}$, where r is a positive constant. For $(\tilde{x}, \tilde{z}) \in Q$, we have

$$\begin{aligned}
 \|\tilde{x}_t + \tilde{\phi}_t\|_{\mathcal{B}} &\leq \|\tilde{x}_t\|_{\mathcal{B}} + \|\tilde{\phi}_t\|_{\mathcal{B}} \\
 &\leq K_b \sup_{0 \leq s \leq t} \|\tilde{x}(s)\| + M_b \|\tilde{x}_0\|_{\mathcal{B}} + K_b \sup_{0 \leq s \leq t} \|\tilde{\phi}(s)\| + M_b \|\tilde{\phi}_0\|_{\mathcal{B}} \\
 &\leq K_b(r + M\|\phi(0)\| + N\|\varphi(0)\|) + M_b\|\phi\|_{\mathcal{B}} = r_1.
 \end{aligned} \tag{2.10}$$

Similarly, we have

$$\begin{aligned}
 \|\tilde{x}'_t + \tilde{\varphi}_t\|_{\mathcal{B}} &\leq \|\tilde{x}'_t\|_{\mathcal{B}} + \|\tilde{\varphi}_t\|_{\mathcal{B}} \\
 &\leq K_b \sup_{0 \leq s \leq t} \|\tilde{x}'(s)\| + M_b \|\tilde{x}'_0\|_{\mathcal{B}} + K_b \sup_{0 \leq s \leq t} \|\tilde{\varphi}(s)\| + M_b \|\tilde{\varphi}_0\|_{\mathcal{B}} \\
 &\leq K_b(r + \tilde{N}\|\phi(0)\| + M\|\varphi(0)\|) + M_b\|\varphi\|_{\mathcal{B}} = r_2,
 \end{aligned} \tag{2.11}$$

where $\tilde{N} = \sup_{t \in J} \|AS(t)\|_{\mathcal{L}(E, X)}$.

For $(\tilde{x}, \tilde{z}) \in Q$, taking norm on (2.8), we obtain

$$\begin{aligned}
 \|\Phi_1(\tilde{x}, \tilde{z})(t)\| &\leq N \int_0^t [k_1\|\tilde{x}_s + \tilde{\phi}_s\|_{\mathcal{B}} + k_2\|\tilde{x}'_s + \tilde{\varphi}_s\|_{\mathcal{B}} + l_f]ds \\
 &+ \frac{1}{\alpha} N^2 K^2 b \{ \|x_1\| + M\|\phi(0)\| + N\|\varphi(0)\| \\
 &+ N \int_0^b [k_1\|\tilde{x}_s + \tilde{\phi}_s\|_{\mathcal{B}} + k_2\|\tilde{x}'_s + \tilde{\varphi}_s\|_{\mathcal{B}} + l_f]ds \\
 &+ M \sum_{k=1}^m [\|I_k^1(\tilde{x}(t_k) + \tilde{\phi}(t_k)) - I_k^1(\tilde{\phi}(t_k))\| + \|I_k^1(\tilde{\phi}(t_k))\|] \\
 &+ N \sum_{k=1}^m [\|I_k^2(\tilde{x}'(t_k^+) + \tilde{\varphi}(t_k^+)) - I_k^2(\tilde{\varphi}(t_k^+))\| + \|I_k^2(\tilde{\varphi}(t_k^+))\|] \\
 &+ M \sum_{k=1}^m [\|I_k^1(\tilde{x}(t_k) + \tilde{\phi}(t_k)) - I_k^1(\tilde{\phi}(t_k))\| + \|I_k^1(\tilde{\phi}(t_k))\|]
 \end{aligned}$$

$$\begin{aligned}
& + N \sum_{k=1}^m [\|I_k^2(\tilde{x}'(t_k^+) + \tilde{\varphi}(t_k^+)) - I_k^2(\tilde{\varphi}(t_k^+))\| + \|I_k^2(\tilde{\varphi}(t_k^+))\|] \\
& \leq Nb\{k_1[K_b(r + M\|\phi(0)\| + N\|\varphi(0)\|) + M_b\|\phi\|_{\mathcal{B}}] \\
& \quad + k_2[K_b(r + \tilde{N}\|\phi(0)\| + M\|\varphi(0)\|) + M_b\|\varphi\|_{\mathcal{B}}] + l_f\} \\
& \quad + \frac{1}{\alpha} N^2 K^2 b \{\|x_1\| + M\|\phi(0)\| + N\|\varphi(0)\| \\
& \quad + Nb\{k_1[K_b(r + M\|\phi(0)\| + N\|\varphi(0)\|) + M_b\|\phi\|_{\mathcal{B}}] \\
& \quad + k_2[K_b(r + \tilde{N}\|\phi(0)\| + M\|\varphi(0)\|) + M_b\|\varphi\|_{\mathcal{B}}] + l_f\} \\
& \quad + M \sum_{k=1}^m [L(I_k^1)r + \|I_k^1(\tilde{\phi}(t_k))\|] + N \sum_{k=1}^m [L(I_k^2)r + \|I_k^2(\tilde{\varphi}(t_k^+))\|] \\
& \quad + M \sum_{k=1}^m [L(I_k^1)r + \|I_k^1(\tilde{\phi}(t_k))\|] + N \sum_{k=1}^m [L(I_k^2)r + \|I_k^2(\tilde{\varphi}(t_k^+))\|] \\
& = NbN_1 + \frac{1}{\alpha} N^2 K^2 b N_2 + N_3,
\end{aligned}$$

where $N_1 = k_1 r_1 + k_2 r_2 + l_f$, $N_2 = \|x_1\| + M\|\phi(0)\| + N\|\varphi(0)\| + NbN_1 + N_3$, $N_3 = M \sum_{k=1}^m [L(I_k^1)r + \|I_k^1(\tilde{\phi}(t_k))\|] + N \sum_{k=1}^m [L(I_k^2)r + \|I_k^2(\tilde{\varphi}(t_k^+))\|]$.

Now let $NbN_1 + \frac{1}{\alpha} N^2 K^2 b N_2 + N_3 < r$, which is equivalent to

$$\begin{aligned}
& Nb\{k_1[K_b(M\|\phi(0)\| + N\|\varphi(0)\|) + M_b\|\phi\|_{\mathcal{B}}] \\
& \quad + k_2[K_b(\tilde{N}\|\phi(0)\| + M\|\varphi(0)\|) + M_b\|\varphi\|_{\mathcal{B}}] + l_f\} \\
& \quad + \frac{1}{\alpha} N^2 K^2 b \{\|x_1\| + M\|\phi(0)\| + N\|\varphi(0)\| \\
& \quad + Nb\{k_1[K_b(M\|\phi(0)\| + N\|\varphi(0)\|) + M_b\|\phi\|_{\mathcal{B}}] \\
& \quad + k_2[K_b(\tilde{N}\|\phi(0)\| + M\|\varphi(0)\|) + M_b\|\varphi\|_{\mathcal{B}}] + l_f\} \\
& \quad + M \sum_{k=1}^m \|I_k^1(\tilde{\phi}(t_k))\| + N \sum_{k=1}^m \|I_k^2(\tilde{\varphi}(t_k^+))\| \\
& \quad + M \sum_{k=1}^m \|I_k^1(\tilde{\phi}(t_k))\| + N \sum_{k=1}^m \|I_k^2(\tilde{\varphi}(t_k^+))\| \\
& < r \{1 - Nb(k_1 + k_2)K_b - M \sum_{k=1}^m L(I_k^1) - N \sum_{k=1}^m L(I_k^2) \\
& \quad - \frac{1}{\alpha} N^2 K^2 b [Nb(k_1 + k_2)K_b - M \sum_{k=1}^m L(I_k^1) - N \sum_{k=1}^m L(I_k^2)]\}. \tag{2.12}
\end{aligned}$$

There exists a r such that (2.12) hold if

$$\phi_1 = (1 + \frac{1}{\alpha} N^2 K^2 b) [Nb(k_1 + k_2)K_b + M \sum_{k=1}^m L(I_k^1) + N \sum_{k=1}^m L(I_k^2)] < 1. \tag{2.13}$$

Similarly, taking norm on (2.9), we obtain

$$\|\Phi_2(\tilde{x}, \tilde{z})(t)\| \leq MbN_1 + \frac{1}{\alpha} MNK^2 b N_2 + N_4,$$

where $N_4 = \tilde{N} \sum_{k=1}^m [L(I_k^1)r + \|I_k^1(\tilde{\phi}(t_k))\|] + M \sum_{k=1}^m [L(I_k^2)r + \|I_k^2(\tilde{\varphi}(t_k^+))\|]$.

Now let $MbN_1 + \frac{1}{\alpha}MNK^2bN_2 + N_4 < r$. Similarly, this is true for large r if

$$\begin{aligned} \phi_2 = & Mb(k_1 + k_2)K_b + \frac{1}{\alpha}MNK^2b[Nb(k_1 + k_2)K_b + M \sum_{k=1}^m L(I_k^1) + N \sum_{k=1}^m L(I_k^2)] \\ & + \tilde{N} \sum_{k=1}^m L(I_k^1) + M \sum_{k=1}^m L(I_k^2) < 1. \end{aligned} \tag{2.14}$$

Therefore, Φ maps Q into Q , when $\max\{\phi_1, \phi_2\} < 1$.

Next, we show that Φ is a contraction mapping on Q . For this, let us take $(\tilde{x}, \tilde{z}), (\tilde{v}, \tilde{w}) \in Q$, then we get

$$\begin{aligned} & \|\Phi_1(\tilde{x}, \tilde{z})(t) - \Phi_1(\tilde{v}, \tilde{w})(t)\| \\ & \leq \int_0^b N(k_1\|\tilde{x}_s - \tilde{v}_s\|_{\mathcal{B}} + k_2\|\tilde{x}'_s - \tilde{v}'_s\|_{\mathcal{B}})ds \\ & \quad + \frac{1}{\alpha}N^2K^2b \left[\int_0^b N(k_1\|\tilde{x}_s - \tilde{v}_s\|_{\mathcal{B}} + k_2\|\tilde{x}'_s - \tilde{v}'_s\|_{\mathcal{B}})ds \right. \\ & \quad + \sum_{k=1}^m ML(I_k^1)\|\tilde{x}(t_k) - \tilde{v}(t_k)\| + \sum_{k=1}^m NL(I_k^2)\|\tilde{x}'(t_k^+) - \tilde{v}'(t_k^+)\| \\ & \quad \left. + \sum_{k=1}^m ML(I_k^1)\|\tilde{x}(t_k) - \tilde{v}(t_k)\| + \sum_{k=1}^m NL(I_k^2)\|\tilde{x}'(t_k^+) - \tilde{v}'(t_k^+)\| \right] \\ & \leq (1 + \frac{1}{\alpha}N^2K^2b) \int_0^b N(k_1\|\tilde{x}_s - \tilde{v}_s\|_{\mathcal{B}} + k_2\|\tilde{x}'_s - \tilde{v}'_s\|_{\mathcal{B}})ds \\ & \quad + (1 + \frac{1}{\alpha}N^2K^2b) \sum_{k=1}^m ML(I_k^1)\|\tilde{x}(t_k) - \tilde{v}(t_k)\| \\ & \quad + (1 + \frac{1}{\alpha}N^2K^2b) \sum_{k=1}^m NL(I_k^2)\|\tilde{x}'(t_k^+) - \tilde{v}'(t_k^+)\|. \end{aligned}$$

In view of

$$\|\tilde{x}_s - \tilde{v}_s\|_{\mathcal{B}} \leq K_b \sup_{0 \leq \tau \leq s} \|\tilde{x}(\tau) - \tilde{v}(\tau)\|$$

and

$$\|\tilde{x}'_s - \tilde{v}'_s\|_{\mathcal{B}} \leq K_b \sup_{0 \leq \tau \leq s} \|\tilde{x}'(\tau) - \tilde{v}'(\tau)\|.$$

We have

$$\begin{aligned} & \|\Phi_1(\tilde{x}, \tilde{z})(t) - \Phi_1(\tilde{v}, \tilde{w})(t)\| \\ & \leq (1 + \frac{1}{\alpha}N^2K^2b)(Nk_1bK_b + \sum_{k=1}^m ML(I_k^1))\|\tilde{x} - \tilde{v}\|_{\mathcal{B}''_{h_1}} \\ & \quad + (1 + \frac{1}{\alpha}N^2K^2b)(Nk_2bK_b + \sum_{k=1}^m NL(I_k^2))\|\tilde{z} - \tilde{w}\|_{\mathcal{B}''_{h_1}}. \end{aligned} \tag{2.15}$$

Similarly, we have

$$\begin{aligned} & \|\Phi_2(\tilde{x}, \tilde{z})(t) - \Phi_2(\tilde{v}, \tilde{w})(t)\| \\ & \leq (M + \frac{1}{\alpha}MN^2K^2b) \int_0^b (k_1\|\tilde{x}_s - \tilde{v}_s\|_{\mathcal{B}} + k_2\|\tilde{x}'_s - \tilde{v}'_s\|_{\mathcal{B}}) ds \\ & \quad + (\frac{1}{\alpha}MNK^2b \sum_{k=1}^m ML(I_k^1) + \sum_{k=1}^m \tilde{N}L(I_k^1)) \|\tilde{x}(t_k) - \tilde{v}(t_k)\| \\ & \quad + (\frac{1}{\alpha}MNK^2b \sum_{k=1}^m NL(I_k^2) + \sum_{k=1}^m ML(I_k^2)) \|\tilde{x}'(t_k^+) - \tilde{v}'(t_k^+)\|. \end{aligned}$$

Finally, we have

$$\begin{aligned} & \|\Phi_2(\tilde{x}, \tilde{z})(t) - \Phi_2(\tilde{v}, \tilde{w})(t)\| \\ & \leq [(M + \eta N)k_1bK_b + \sum_{k=1}^m (\eta M + \tilde{N})L(I_k^1)] \|\tilde{x} - \tilde{v}\|_{\mathcal{B}'_{h_1}} \\ & \quad + [(M + \eta N)k_2bK_b + \sum_{k=1}^m (\eta N + M)L(I_k^2)] \|\tilde{z} - \tilde{w}\|_{\mathcal{B}'_{h_1}}. \end{aligned} \quad (2.16)$$

The above inequalities (2.15) and (2.16) and the assumption $\max\{\phi_1, \phi_2\} < 1$ imply that Φ is a contraction mapping. Hence there exists a unique fixed point $(\tilde{x}, \tilde{z}) \in Q$. Then the function $x(\cdot) = \tilde{x}(\cdot) + \tilde{\phi}(\cdot) \in \mathcal{B}'_{h_2}$ is a mild solution of (1.1). This completes the proof. \square

3. Approximate Controllability

In this section, we compare approximate controllability of the semilinear system (1.1) with approximate controllability of the associated linear system. For this reason, we consider the linear system

$$x''(t) = Ax(t) + Bu(t), \quad t \in J, \quad (3.1)$$

with initial condition

$$x(0) = \phi(0), \quad x'(0) = \varphi(0). \quad (3.2)$$

Both the exact and the approximate controllability of systems (3.1)-(3.2) have been studied by several authors. Directly related to systems modeled by (3.1)-(3.2), we mention the works [6, 7, 9, 11, 17, 18, 20, 27, 29, 30].

Definition 3.1. Systems (3.1)-(3.2) are said to be approximately controllable on J if $\overline{D} = X \times X$, where $D = \{x(b, \phi(0), \varphi(0), u), y(b, \phi(0), \varphi(0), u) : u \in L^2(J, U)\}$, $y(\cdot, \phi(0), \varphi(0), u) = x'(\cdot, \phi(0), \varphi(0), u)$ and $x(\cdot, \phi(0), \varphi(0), u)$ is a mild solution of (3.1)-(3.2).

The following result has been established by Fattorini [7] and Triggiani [27, 28].

We introduce the sets

$$D_\infty(A) = \cap_{n=1}^\infty D(A^n),$$

$$U_\infty = \{u \in U : Bu \in D_\infty(A)\},$$

$$X_0 = \cup_{t>0} T(t)(X),$$

$$U_0 = \{u \in U : Bu \in X_0\},$$

where $T(t)$ is the analytic semigroup generated by A [1,8]. It is clear that $U_0 \subseteq U_\infty$.

Theorem 3.1 (see [7,27,28]).

(a) Systems (3.1)-(3.2) are approximately controllable on J if, and only if, $x^*, y^* \in X^*$ are such that $B^*S(t)x^* + B^*C(t)^*y^* = 0$, for $t \in J$, then $x^* = y^* = 0$.

(b) If $Sp\{A^nBU_\infty : n \geq 0\}$ is dense in X , then systems (3.1)-(3.2) are approximately controllable on J .

(c) If BU_0 is dense in BU and systems (3.1)-(3.2) are approximately controllable on J , then $Sp\{A^nBU_0 : n \geq 0\}$ is dense in X .

We return to the controllability problem for the semilinear system (1.1). Before stating and proving our main result, we give first the definition of approximate controllability.

Definition 3.2. System (1.1) is said to be approximately controllable on J if $\overline{\mathcal{R}(f, \phi, \varphi)} = X \times X$, where $\mathcal{R}(f, \phi, \varphi) = \{x(b, \phi, \varphi, u), y(b, \phi, \varphi, u) : u \in L^2(J, U)\}$, $y(\cdot, \phi, \varphi, u) = x'(\cdot, \phi, \varphi, u)$ and $x(\cdot, \phi, \varphi, u)$ is a mild solution of (1.1).

Theorem 3.2. Assume that BU_0 is dense in BU and the conditions (H_2) - (H_4) are satisfied. If systems (3.1)-(3.2) are approximately controllable on J , then system (1.1) is approximately controllable on J .

Proof. It follows by the approximately controllability of (3.1)-(3.2) on J , we obtain (H_1) is satisfied. Because the hypotheses of Theorem 2.1 are fulfilled, for each $u \in L^2(J, U)$, there is a unique mild solution of (1.1). Let (\tilde{x}, \tilde{z}) be a fixed point of Φ in Q . $x(\cdot) = \tilde{x}(\cdot) + \tilde{\phi}(\cdot)$ is the mild solution of (1.1) on J . By the conditions (H_2) and the proof of Theorem 2.1

$$\|f(s, x_s, x'_s)\| \leq k_1r_1 + k_2r_2 + l_f.$$

We fix $z = (z_1, z_2) \in X \times X$. We take $0 < b_n < b$ such that $b_n \rightarrow b$ as $n \rightarrow \infty$. Let $x_n = x(b_n, \phi, \varphi, 0)$ and $y_n = y(b_n, \phi, \varphi, 0)$. It follows from the properties established in Section 2 that $x_n \in E$. In addition, it follows from Theorem 3.1 that system (3.1) with initial conditions $x(0) = x_n$ and $x'(0) = y_n$ is approximately controllable on $[0, b - b_n]$. Consequently, there is a control function $w_n(\cdot) \in L^p([0, b - b_n], U)$ such that

$$\begin{aligned} & \int_0^{b-b_n} S(b-b_n-s)Bw_n(s)ds + C(b-b_n)x_n + S(b-b_n)y_n - z_1 \\ &= \int_{b_n}^b S(b-s)Bv_n(s)ds + C(b-b_n)x_n + S(b-b_n)y_n - z_1 \rightarrow 0, \quad n \rightarrow \infty, \end{aligned}$$

and

$$\begin{aligned} & \int_0^{b-b_n} C(b-b_n-s)Bw_n(s)ds + AS(b-b_n)x_n + C(b-b_n)y_n - z_2 \\ &= \int_{b_n}^b C(b-s)Bv_n(s)ds + AS(b-b_n)x_n + C(b-b_n)y_n - z_2 \rightarrow 0, \quad n \rightarrow \infty, \end{aligned}$$

where $v_n(s) = w_n(s - b_n)$. We define

$$u_n(s) = \begin{cases} 0, & 0 \leq s \leq b_n, \\ v_n(s), & b_n < s \leq b. \end{cases}$$

In the following development, we use the abbreviate notation $x(\cdot) = x(\cdot, \phi, \varphi, u_n)$ and $y(\cdot) = y(\cdot, \phi, \varphi, u_n)$. Using the uniqueness of solutions, we have that

$$\begin{aligned} x_n &= C(b_n)\phi(0) + S(b_n)\varphi(0) + \int_0^{b_n} S(b_n-s)f(s, x_s, x'_s)ds \\ &\quad + \sum_{0 < t_k < b_n} C(b_n-t_k)I_k^1(x(t_k)) + \sum_{0 < t_k < b_n} S(b_n-t_k)I_k^2(x'(t_k^+)), \\ y_n &= AS(b_n)\phi(0) + C(b_n)\varphi(0) + \int_0^{b_n} C(b_n-s)f(s, x_s, x'_s)ds \\ &\quad + \sum_{0 < t_k < b_n} AS(b_n-t_k)I_k^1(x(t_k)) + \sum_{0 < t_k < b_n} C(b_n-t_k)I_k^2(x'(t_k^+)). \end{aligned}$$

Combining these expressions with (2.1) and (2.2), we obtain

$$\begin{aligned} x(b, \phi, \varphi, u_n) &= C(b)\phi(0) + S(b)\varphi(0) + \int_0^b S(b-s)Bu_n(s)ds \\ &\quad + \int_0^b S(b-s)f(s, x_s, x'_s)ds \\ &\quad + \sum_{k=1}^m C(b-t_k)I_k^1(x(t_k)) + \sum_{k=1}^m S(b-t_k)I_k^2(x'(t_k^+)) \\ &= C(b)\phi(0) + S(b)\varphi(0) + \int_{b_n}^b S(b-s)Bv_n(s)ds \\ &\quad + \int_0^{b_n} S(b-s)f(s, x_s, x'_s)ds + \int_{b_n}^b S(b-s)f(s, x_s, x'_s)ds \\ &\quad + \sum_{k=1}^m C(b-t_k)I_k^1(x(t_k)) + \sum_{k=1}^m S(b-t_k)I_k^2(x'(t_k^+)) \\ &= C(b)\phi(0) + S(b)\varphi(0) + \int_{b_n}^b S(b-s)Bv_n(s)ds \\ &\quad + S(b-b_n) \int_0^{b_n} C(b_n-s)f(s, x_s, x'_s)ds \\ &\quad + C(b-b_n) \int_0^{b_n} S(b_n-s)f(s, x_s, x'_s)ds \end{aligned}$$

$$\begin{aligned}
 & + \int_{b_n}^b S(b-s)f(s, x_s, x'_s)ds \\
 & + \sum_{k=1}^m C(b-t_k)I_k^1(x(t_k)) + \sum_{k=1}^m S(b-t_k)I_k^2(x'(t_k^+)) \\
 = & C(b)\phi(0) + S(b)\varphi(0) + \int_{b_n}^b S(b-s)Bv_n(s)ds \\
 & + \int_{b_n}^b S(b-s)f(s, x_s, x'_s)ds \\
 & + \sum_{k=1}^m C(b-t_k)I_k^1(x(t_k)) + \sum_{k=1}^m S(b-t_k)I_k^2(x'(t_k^+)) \\
 & + S(b-b_n)[y_n - AS(b_n)\phi(0) - C(b_n)\varphi(0) \\
 & - \sum_{0 < t_k < b_n} AS(b_n-t_k)I_k^1(x(t_k)) - \sum_{0 < t_k < b_n} C(b_n-t_k)I_k^2(x'(t_k^+))] \\
 & + C(b-b_n)[x_n - C(b_n)\phi(0) - S(b_n)\varphi(0) \\
 & - \sum_{0 < t_k < b_n} C(b_n-t_k)I_k^1(x(t_k)) - \sum_{0 < t_k < b_n} S(b_n-t_k)I_k^2(x'(t_k^+))] \\
 = & \int_{b_n}^b S(b-s)Bv_n(s)ds + \int_{b_n}^b S(b-s)f(s, x_s, x'_s)ds \\
 & + \sum_{k=1}^m C(b-t_k)I_k^1(x(t_k)) + \sum_{k=1}^m S(b-t_k)I_k^2(x'(t_k^+)) \\
 & + S(b-b_n)[y_n - \sum_{0 < t_k < b_n} AS(b_n-t_k)I_k^1(x(t_k)) \\
 & - \sum_{0 < t_k < b_n} C(b_n-t_k)I_k^2(x'(t_k^+))] \\
 & + C(b-b_n)[x_n - \sum_{0 < t_k < b_n} C(b_n-t_k)I_k^1(x(t_k)) \\
 & - \sum_{0 < t_k < b_n} S(b_n-t_k)I_k^2(x'(t_k^+))].
 \end{aligned}$$

Because the function f is bounded on Q , we infer that

$$\int_{b_n}^b S(b-s)f(s, x_s, x'_s)ds \rightarrow 0, \quad n \rightarrow \infty.$$

In addition, from (2.1) and (2.2), all the summation term cancel. Thus we obtain $x(b, \phi, \varphi, u_n) \rightarrow z_1$ as $n \rightarrow \infty$.

In a similiar way

$$\begin{aligned}
 y(b, \phi, \varphi, u_n) = & AS(b)\phi(0) + C(b)\varphi(0) + \int_0^b C(b-s)Bu_n(s)ds \\
 & + \int_0^b C(b-s)f(s, x_s, x'_s)ds
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^m AS(b-t_k)I_k^1(x(t_k)) + \sum_{k=1}^m C(b-t_k)I_k^2(x'(t_k^+)) \\
= & AS(b)\phi(0) + C(b)\varphi(0) + \int_{b_n}^b C(b-s)Bv_n(s)ds \\
& + \int_{b_n}^b C(b-s)f(s, x_s, x'_s)ds \\
& + \sum_{k=1}^m AS(b-t_k)I_k^1(x(t_k)) + \sum_{k=1}^m C(b-t_k)I_k^2(x'(t_k^+)) \\
& + C(b-b_n)[y_n - AS(b_n)\phi(0) - C(b_n)\varphi(0) \\
& - \sum_{0 < t_k < b_n} AS(b_n-t_k)I_k^1(x(t_k)) - \sum_{0 < t_k < b_n} C(b_n-t_k)I_k^2(x'(t_k^+))] \\
& + AS(b-b_n)[x_n - C(b_n)\phi(0) - S(b_n)\varphi(0) \\
& - \sum_{0 < t_k < b_n} C(b_n-t_k)I_k^1(x(t_k)) - \sum_{0 < t_k < b_n} S(b_n-t_k)I_k^2(x'(t_k^+))] \\
= & \int_{b_n}^b C(b-s)Bv_n(s)ds + \int_{b_n}^b C(b-s)f(s, x_s, x'_s)ds \\
& + \sum_{k=1}^m AS(b-t_k)I_k^1(x(t_k)) + \sum_{k=1}^m C(b-t_k)I_k^2(x'(t_k^+)) \\
& + C(b-b_n)[y_n - \sum_{0 < t_k < b_n} AS(b_n-t_k)I_k^1(x(t_k)) \\
& - \sum_{0 < t_k < b_n} C(b_n-t_k)I_k^2(x'(t_k^+))] \\
& + AS(b-b_n)[x_n - \sum_{0 < t_k < b_n} C(b_n-t_k)I_k^1(x(t_k)) \\
& - \sum_{0 < t_k < b_n} S(b_n-t_k)I_k^2(x'(t_k^+))].
\end{aligned}$$

Using again that f is bounded, we infer that $\int_{b_n}^b C(b-s)f(s, x_s, x'_s)ds \rightarrow 0$, $n \rightarrow \infty$. Again, as in $x(b, \phi, \varphi, u_n)$, from (2.1) and (2.2), all the summation terms cancel as $n \rightarrow \infty$. Thus, $y(b, \phi, \varphi, u_n) \rightarrow z_2$ as $n \rightarrow \infty$.

This implies that $z \in \overline{\mathcal{R}(f, \phi, \varphi)}$. Because z was arbitrarily chosen, this completes the proof. \square

4. Example

In this section we present an example of controllable impulsive partial differential equation with infinite delay. In the following, $X = L^2([0, \pi])$; $\mathcal{B} = \mathcal{PC}_0 \times L^2(\rho, X)$ and $A : D(A) \subseteq X \rightarrow X$ is the map defined by $Af = f''$ with domain $D(A) = \{f \in X : f \text{ and } f' \text{ are absolutely continuous, } f'' \in X, f(0) = f(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a strongly continuous cosine family of operators, $\{C(t) : t \in R\}$ on X . Furthermore, A has a discrete spectrum and the eigenvalues are $-n^2, n \in N$, with corresponding normalized eigenvectors

$z_n(\xi) = \sqrt{\frac{2}{\pi}} \sin(n\xi)$; the set $\{z_n, n \in N\}$ is an orthonormal basis of X and the following properties hold:

- (a) If $f \in D(A)$, then $Af = - \sum_{n=1}^{\infty} n^2 \langle f, z_n \rangle z_n$.
- (b) For $f \in X$, $C(t)f = \sum_{n=1}^{\infty} \cos(nt) \langle f, z_n \rangle z_n$ and $S(t)f = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle f, z_n \rangle z_n$.
Consequently, $\|C(t)\| = \|S(t)\| \leq 1$ for all $t \in R$.
- (c) If Φ is the group of translations on X defined by $\Phi(t)x(\xi) = \tilde{x}(\xi + t)$, where \tilde{x} is the extension of x with period 2π , then $C(t) = \frac{1}{2}(\Phi(t) + \Phi(-t))$ and $A = B^2$, where B is the generator of Φ and $E = \{x \in H^1([0, \pi]) : x(0) = x(\pi) = 0\}$ (see [8] for details).

Consider the impulsive partial differential equation

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial z(t, x)}{\partial t} \right) &= \frac{\partial^2 z(t, x)}{\partial x^2} + q(x)u(t) \\ &+ \int_{-\infty}^t c(t-s) \left(z(s, x) + \frac{\partial z(s, x)}{\partial s} \right) ds, \quad x \in [0, \pi], \quad t \in J, \quad t \neq t_k, \\ z(t, 0) &= z(t, \pi) = 0, \quad t \in J, \\ \Delta z(t_k)(x) &= \int_0^\pi K_1(t_k, x, y) z(t_k, y) dy, \quad k = 1, 2, \dots, m, \\ \Delta z'(t_k)(x) &= \int_0^\pi K_2(t_k, x, y) z(t_k, y) dy, \quad k = 1, 2, \dots, m, \\ z(t, x) &= \phi(t, x), \quad t \in (-\infty, 0], \quad x \in [0, \pi], \\ \frac{\partial}{\partial t} z(t, x) &= \varphi(t, x), \quad t \in (-\infty, 0], \quad x \in [0, \pi], \end{aligned} \tag{4.1}$$

where we assume $\phi, \varphi \in \mathcal{B}$, with the identifications $\phi(t)(x) = \phi(t, x)$, $\varphi(t)(x) = \varphi(t, x)$. $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = b$. We assume that the function q can be expressed in the form $q = \sum_{n=1}^{\infty} e^{-n^2} q_n z_n$, where $q_n \neq 0$ for all $n \in N$ and $\sum_{n=1}^{\infty} q_n^2 < \infty$.

We define $B : R \rightarrow X$ by $Bu = qu$. Then $\|B\| \leq K = \sqrt{\sum_{n=1}^{\infty} e^{-2n^2} q_n^2}$. To study the approximate controllability of (4.1), assume $c(t)$ is measurable and continuous with finite $L_f = \left(\int_{-\infty}^0 \frac{c^2(-\theta)}{\rho(\theta)} d\theta \right)^{\frac{1}{2}}$. $K_i(t, x, y) : J \rightarrow L^2(\Delta)$, $\Delta = [0, \pi] \times [0, \pi]$, $l_k^i := \left(\int_0^\pi \int_0^\pi |K_i(t_k, x, y)|^2 dx dy \right)^{\frac{1}{2}}$, $i = 1, 2, k = 1, 2, \dots, m$.

Defining the operator $f : J \times \mathcal{B} \times \mathcal{B} \rightarrow X$ by

$$f(t, \phi, \varphi)(x) = \int_{-\infty}^0 c(-\theta) (\phi(\theta, x) + \varphi(\theta, x)) d\theta.$$

Also defining the maps I_k^1 and I_k^2

$$\begin{aligned} I_k^1(w)(x) &= \int_0^\pi K_1(t_k, x, y) w(y) dy, \quad w \in X, \\ I_k^2(w)(x) &= \int_0^\pi K_2(t_k, x, y) w(y) dy, \quad w \in X, \end{aligned}$$

then system (4.1) can be modelled as (1.1).

Define

$$\Gamma_t^b = \int_t^b S(b-s)BB^*S^*(b-s)ds.$$

We claim that $B^*S^*(b-s)x^* + B^*C^*(b-s)y^* = 0$, $0 \leq s \leq b$ implies that $x^* = y^* = 0$. Indeed

$$\begin{aligned} & B^*S^*(b-s)x^* + B^*C^*(b-s)y^* = 0, \quad 0 \leq s \leq b \\ \implies & \sum_{n=1}^{\infty} e^{-n^2} q_n \langle z_n, \sum_{k=1}^{\infty} \frac{\sin k(b-s)}{k} \langle x^*, z_k \rangle z_k \rangle \\ & + \sum_{n=1}^{\infty} e^{-n^2} q_n \langle z_n, \sum_{k=1}^{\infty} \cos k(b-s) \langle y^*, z_k \rangle z_k \rangle = 0 \\ \implies & \sum_{n=1}^{\infty} e^{-n^2} q_n \frac{\sin n(b-s)}{n} \langle x^*, z_n \rangle \\ & + \sum_{n=1}^{\infty} e^{-n^2} q_n \cos n(b-s) \langle y^*, z_n \rangle = 0 \\ \implies & x^* = y^* = 0. \end{aligned}$$

It follows from Theorem 3.1 that the linear systems (3.1)-(3.2) are approximately controllable on J . Then the operator $\alpha(\alpha I + \Gamma_t^b)^{-1} \rightarrow 0$ in the strong operator topology as $\alpha \rightarrow 0^+$ (see [4, 27, 28]). So assumption (H_1) is satisfied.

In these conditions

$$\begin{aligned} & |f(t, \phi, \varphi)|_{L^2}^2 \\ &= \int_0^\pi \left[\int_{-\infty}^0 c(-\theta)(\phi(\theta, x) + \varphi(\theta, x))d\theta \right]^2 dx \\ &\leq 2 \int_0^\pi \left[\int_{-\infty}^0 c(-\theta)\phi(\theta, x)d\theta \right]^2 dx + 2 \int_0^\pi \left[\int_{-\infty}^0 c(-\theta)\varphi(\theta, x)d\theta \right]^2 dx \\ &= 2 \int_0^\pi \left[\int_{-\infty}^0 \frac{c(-\theta)}{\rho^{\frac{1}{2}}(\theta)} \rho^{\frac{1}{2}}(\theta)\phi(\theta, x)d\theta \right]^2 dx + 2 \int_0^\pi \left[\int_{-\infty}^0 \frac{c(-\theta)}{\rho^{\frac{1}{2}}(\theta)} \rho^{\frac{1}{2}}(\theta)\varphi(\theta, x)d\theta \right]^2 dx \\ &\leq 2 \int_0^\pi \left[\int_{-\infty}^0 \frac{c^2(-\theta)}{\rho(\theta)} d\theta \cdot \int_{-\infty}^0 \rho(\theta)|\phi(\theta, x)|^2 d\theta \right] dx \\ &\quad + 2 \int_0^\pi \left[\int_{-\infty}^0 \frac{c^2(-\theta)}{\rho(\theta)} d\theta \cdot \int_{-\infty}^0 \rho(\theta)|\varphi(\theta, x)|^2 d\theta \right] dx \\ &= 2 \int_{-\infty}^0 \frac{c^2(-\theta)}{\rho(\theta)} d\theta \cdot \left[\int_{-\infty}^0 \rho(\theta) \int_0^\pi |\phi(\theta, x)|^2 dx d\theta \right] \\ &\quad + 2 \int_{-\infty}^0 \frac{c^2(-\theta)}{\rho(\theta)} d\theta \cdot \left[\int_{-\infty}^0 \rho(\theta) \int_0^\pi |\varphi(\theta, x)|^2 dx d\theta \right]. \\ &\leq 2L_f^2 [\|\phi\|_{\mathcal{B}}^2 + \|\varphi\|_{\mathcal{B}}^2], \end{aligned}$$

which implies that the function f verifies the following condition

$$\begin{aligned} & |f(t, \phi_1, \varphi_1) - f(t, \phi_2, \varphi_2)|_{L^2} \\ &= \left\{ \int_0^\pi \left[\int_{-\infty}^0 c(-\theta)(\phi_1(\theta, x) + \varphi_1(\theta, x))d\theta \right. \right. \\ &\quad \left. \left. - \int_{-\infty}^0 c(-\theta)(\phi_2(\theta, x) + \varphi_2(\theta, x))d\theta \right]^2 dx \right\}^{1/2} \\ &= \left\{ \int_0^\pi \left[\int_{-\infty}^0 c(-\theta)(\phi_1(\theta, x) - \phi_2(\theta, x))d\theta \right. \right. \\ &\quad \left. \left. + \int_{-\infty}^0 c(-\theta)(\varphi_1(\theta, x) - \varphi_2(\theta, x))d\theta \right]^2 dx \right\}^{1/2} \\ &\leq \sqrt{2L_f^2 [\|\phi_1 - \phi_2\|_{\mathcal{B}}^2 + \|\varphi_1 - \varphi_2\|_{\mathcal{B}}^2]} \\ &\leq \sqrt{2}L_f (\|\phi_1 - \phi_2\|_{\mathcal{B}} + \|\varphi_1 - \varphi_2\|_{\mathcal{B}}), \end{aligned}$$

which means (H_2) is satisfied.

In a similar way

$$\begin{aligned} & |I_k^i(w_1) - I_k^i(w_2)|_{L^2} \\ &= \left\{ \int_0^\pi \left[\int_0^\pi K_i(t_k, x, y)w_1(y)dy - \int_0^\pi K_i(t_k, x, y)w_2(y)dy \right]^2 dx \right\}^{1/2} \\ &= \left\{ \int_0^\pi \left[\int_0^\pi K_i(t_k, x, y)(w_1(y) - w_2(y))dy \right]^2 dx \right\}^{1/2} \\ &\leq \left\{ \int_0^\pi \left[\int_0^\pi |K_i(t_k, x, y)|^2 dy \cdot \int_0^\pi |w_1(y) - w_2(y)|^2 dy \right] dx \right\}^{1/2} \\ &\leq \sqrt{\pi}l_k^i |w_1 - w_2|_{L^2}, \quad i = 1, 2, \end{aligned}$$

which means (H_3) is satisfied.

Moreover, the function $t \rightarrow AS(t)$ is uniformly continuous into $\mathcal{L}(E, X)$ and $\|AS(t)\|_{\mathcal{L}(E, X)} \leq 1$ for $t \in J$.

Let

$$\phi_1 = \left(1 + \frac{1}{\alpha}K^2b\right) \left\{2\sqrt{2}bL_f \left[1 + \left(\int_{-b}^0 \rho(\theta)d\theta\right)^{1/2}\right] + \sqrt{\pi} \sum_{k=1}^m (l_k^1 + l_k^2)\right\}$$

and

$$\begin{aligned} \phi_2 &= 2\sqrt{2}bL_f \left[1 + \left(\int_{-b}^0 \rho(\theta)d\theta\right)^{1/2}\right] + \frac{1}{\alpha}K^2b \left\{2\sqrt{2}bL_f \left[1 + \left(\int_{-b}^0 \rho(\theta)d\theta\right)^{1/2}\right] \right. \\ &\quad \left. + \sqrt{\pi} \sum_{k=1}^m (l_k^1 + l_k^2)\right\} + \sqrt{\pi} \sum_{k=1}^m (l_k^1 + l_k^2). \end{aligned}$$

The next proposition is a consequence of Theorem 3.2.

Proposition 4.1. *Assume $\max\{\phi_1, \phi_2\} < 1$. Then the system (4.1) is approximate controllability.*

5. Conclusion

In this paper, the issue on the approximate controllability criteria for a class of second order impulsive functional differential systems with infinite delay has been addressed for the first time. A new set of sufficient conditions for the approximate controllability of the considered nonlinear systems have been established by using strongly continuous cosine families of operators and the contraction mapping principle. Particularly, we have shown that under the assumption that the approximate controllability of its linear part, this system is approximate controllable. Moreover, the example presented in Section 4 illustrated exactly an application of the obtained results.

The neutral functional differential system in the form of

$$\begin{aligned} \frac{d}{dt}[x'(t) - g(t, x_t)] &= Ax(t) + f(t, x(t)) + Bu(t), \quad t \in [0, T], \\ x_0 = \varphi \in \mathcal{B}, \quad x'(0) &= w \in X, \end{aligned}$$

which was studied in [13], is a special case of our system. So our results are applicable to such system.

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