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调和 Dirichlet 空间上 Toeplitz 算子与小 Hankel 算子的交换性

冯丽霞 [1,2], 麻文芳 [3], 赵连阔 [2]

(1. 西北大学数学与科学史研究中心, 陕西 西安 710127)
(2. 山西师范大学数学与计算机科学学院, 山西 临汾 041004)
(3. 朔州市第一中学, 山西 朔州 036000)

摘 要: 本文研究了调和 Dirichlet 空间上调和符号的 Toeplitz 算子与小 Hankel 算子交换性的问题. 利用算子矩阵表示的方法, 获得了调和 Dirichlet 空间上调和符号的 Toeplitz 算子与小 Hankel 算子交换的充要条件, 将 Dirichlet 空间上的相应结果推广到了调和 Dirichlet 空间上.

关键词: 调和 Dirichlet 空间; Toeplitz 算子; 小 Hankel 算子; 交换性
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1 引言

最近几年, 单圆盘 Dirichlet 空间上的 Toeplitz 算子得到深入研究, 取得了丰富的成果 [1−8]. 同时调和 Dirichlet 空间上 Toeplitz 算子的研究也受到关注 [9−12]. 文 [10, 11] 分别给出了调和 Dirichlet 空间上调和符号 Toeplitz 算子的 (半) 交换性与乘积为零的刻画. 文 [12] 推广了 [10, 11] 中的结论, 给出了一般符号 Toeplitz 算子一些代数性质及紧性的刻画. 本文, 我们考虑调和 Dirichlet 空间上调和符号的 Toeplitz 算子与小 Hankel 算子的交换性.

首先我们回顾调和 Dirichlet 空间及其上 Toeplitz 算子和小 Hankel 算子的基本知识.

设 $\mathbb{D}$ 是复平面 $\mathbb{C}$ 上的单位开圆盘, dA 表示 $\mathbb{D}$ 上正规化的面积测度. 称 $\mathbb{D}$ 上光滑函数 f 在如下范数

$$\|f\|=\left\{\left|\int_{\mathbb{D}} f dA\right|^2+\int_{\mathbb{D}}\left(\left|\frac{\partial f}{\partial z}\right|^2+\left|\frac{\partial f}{\partial \bar{z}}\right|^2\right) dA\right\}^{\frac{1}{2}}<\infty,$$

取闭包所得到的空间为 Sobolev 空间, 记为 $\mathcal{S}$. 则 $\mathcal{S}$ 是一个 Hilbert 空间, 其上内积为

$$\langle f, g\rangle=\int_{\mathbb{D}} f dA \int_{\mathbb{D}} \bar{g} dA+\int_{\mathbb{D}}\left(\frac{\partial f}{\partial z} \overline{\frac{\partial g}{\partial z}}+\frac{\partial f}{\partial \bar{z}} \overline{\frac{\partial g}{\partial \bar{z}}}\right) dA, \quad f, g \in S.$$

Dirichlet 空间是由 $\mathcal{S}$ 中所有满足条件 $f(0)=0$ 的解析函数 f 构成的闭子空间, 记为 $\mathcal{D}$. $\mathcal{D}$ 的一组标准正交基为 $\{\frac{z^n}{\sqrt{n}}\}_{n=1}^{\infty}$.

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作者简介: 冯丽霞 (1978 –), 女, 山西闻喜, 讲师, 主要研究方向: 泛函分析, 数学史与数学教育.

$\mathcal{S}$ 中所有调和函数构成的闭子空间称为调和 Dirichlet 空间, 记为 $\mathcal{D}_h$. 显然有

$$\mathcal{D}_h = \mathcal{D} \oplus \mathbb{C} \oplus \overline{\mathcal{D}},$$

其中 $\overline{\mathcal{D}} = \{\bar{f} \ \mid \ f \in \mathcal{D}\}$. $\mathcal{D}_h$ 的一组标准正交基为 $\{\frac{z^n}{\sqrt{n}}\}_{n=1}^{\infty} \cup \{1\} \cup \{\frac{\bar{z}^n}{\sqrt{n}}\}_{n=1}^{\infty}$.

$\mathbb{D}$ 上的一个非负测度 μ 称为 $\mathcal{D}$-Carleson 测度如果存在非负常数 c, 使得

$$\int_{\mathbb{D}} |f|^2 d\mu \leq c\|f\|^2, \ f \in \mathcal{D}.$$

$\mathbb{D}$ 上所有有界解析函数全体记为 $H^\infty(\mathbb{D})$. 令

$$\mathcal{M} = \left\{ u \text{ 是 } \mathbb{D}\text{上的调和函数} \ \middle| \ \begin{array}{c} u = f + \bar{g}, \quad f, g \in H^\infty(\mathbb{D}), \\ |f'|^2 dA, \ |g'|^2 dA \text{ 是 } \mathcal{D} - \text{Carleson 测度} \end{array} \right\}.$$

若 $\phi \in \mathcal{M}$, $\mathcal{D}_h$ 上的 Toeplitz 算子 $\widetilde{T}_\phi$ 定义为

$$\widetilde{T}_\phi(f) = Q(\phi f), \quad f \in \mathcal{D}_h,$$

其中 Q 是从 S 到 $\mathcal{D}_h$ 上的正交投影. $\widetilde{T}_\phi$ 是 $\mathcal{D}_h$ 上的有界算子 [12].

定义 $\mathcal{S}$ 上的算子 $U: (Uf)(z) = f(\bar{z})$, $f \in \mathcal{S}$. 则 U 是 $\mathcal{S}$ 上的酉算子, $U^* = U = U^{-1}$ [9].

对于 $\phi \in \mathcal{M}$, 定义 $\mathcal{D}_h$ 上的小 Hankel 算子 $\widetilde{\Gamma}_\phi$ 为

$$\widetilde{\Gamma}_\phi(f) = Q(U(\phi f)), \quad f \in \mathcal{D}_h.$$

容易验证 $\widetilde{\Gamma}_\phi$ 是有界算子. 实际上, 如果记 $\hat{f}(z) = f(\bar{z})$, $f \in \mathcal{S}$, 则有 $\widetilde{\Gamma}_\phi = \widetilde{T}_{\hat{\phi}} U$.

在 Hardy 空间上, 文 [13, 14] 分别给出了 Toeplitz 算子与 Hankel 算子交换性和本质交换性的刻画. 文 [8] 将 Dirichlet 空间上调和符号 Toeplitz 算子与小 Hankel 算子的交换性转化为 Hardy 空间上相应问题的研究. 但目前关于调和函数空间上 Toeplitz 算子与 (小) Hankel 算子的交换性研究较少. 文 [15, 16] 研究了调和 Bergman 空间上 Toeplitz 算子的交换性.

利用 Sobolev 空间的正交分解, 文 [7] 和 [12] 分别刻画了 Dirichlet 空间和调和 Dirichlet 空间上一般符号 Toeplitz 算子的代数性质等, 其证明过程表明, 在 Dirichlet 空间和调和 Dirichlet 空间上一般符号定义的 Toeplitz 算子与调和符号定义的 Toeplitz 算子有着密切的联系, 一般符号 Toeplitz 算子的研究可转化为相应调和符号 Toeplitz 算子的研究上. 因此本文集中研究调和符号定义的 Toeplitz 算子和 Hankel 算子. 主要结论如下.

定理 1.1 设 $\varphi, \psi \in \mathcal{M}$. 在 $\mathcal{D}_h$ 上 $\widetilde{T}_\varphi \widetilde{\Gamma}_\psi = \widetilde{\Gamma}_\psi \widetilde{T}_\varphi$ 当且仅当下列条件之一成立:

1° φ 是常数;

2° φ 不是常数, 存在常数 α, β 使得 $\psi = \alpha\varphi + \beta$, 且当 $\alpha \neq 0$ 或 $\beta \neq 0$ 时, $\varphi = U\varphi$.

2 主要结论的证明

本节, 我们给出定理 1.1 的证明.

设 P 是从 S 到 $\mathcal{D}$ 上的正交投影, P_1 是从 S 到 $\bar{\mathcal{D}}$ 上的正交投影, 则 $P_1 = UPU$ [9]. 对于从 S 到 $\mathcal{D}_h$ 上的正交投影 Q, 我们有下面的结论.

引理 2.1 $UQ = QU$. 即 $\forall \ f \in \mathcal{S}$, $(UQ)f = (QU)f$.

证 设 $f \in S$, 则

$$Q(f) = P(f) + \langle f, 1 \rangle + P_1(f).$$

因此

$$(UQ)f = UPf + U\langle f, 1 \rangle + UP_1 f = UPf + \langle f, 1 \rangle + PUf,$$
$$(QU)f = PUf + \langle Uf, 1 \rangle + P_1 Uf = PUf + \langle \hat{f}, 1 \rangle + UPf.$$

因为

$$\langle f, 1 \rangle = \int_{\mathbb{D}} f(z) dA(z) = \int_{\mathbb{D}} f(\bar{z}) dA(z) = \langle \hat{f}, 1 \rangle,$$

所以由上面两个等式可得 $(UQ)f = (QU)f$, $f \in S$. 证毕.

容易验证 U 将 $\overline{\mathcal{D}}$ 映到 $\mathcal{D}$ 上. 定义算子 $\widetilde{U} : \mathcal{D}_h = \mathcal{D} \oplus \mathbb{C} \oplus \overline{\mathcal{D}} \to \mathcal{D} \oplus \mathbb{C} \oplus \mathcal{D}$, 其中

$$\tilde{U} = \begin{pmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & U \end{pmatrix}.$$

显然, $\tilde{U}^*$ 将 $\mathcal{D} \oplus \mathbb{C} \oplus \mathcal{D}$ 映到 $\mathcal{D}_h$ 上, 且

$$\tilde{U}^* = \begin{pmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & U \end{pmatrix}.$$

设 $\phi \in \mathcal{M}$, 定义

$$(1 \otimes \phi)(f) = \int_{\mathbb{D}} f(z)\phi(z) dA(z),\ f \in \mathcal{D}_h, P_\phi(a) = aP(\phi),\ a \in \mathbb{C}.$$

T_ϕ 表示 $\mathcal{D}$ 上的 Toeplitz 算子, $T_\phi(f) = P(\phi f),\ f \in \mathcal{D}$. $\varGamma_\phi$ 表示 $\mathcal{D}$ 上的小 Hankel 算子

$$\varGamma_\phi(f) = P(U(\phi f)), \quad f \in \mathcal{D}.$$

下面的引理分别给出了调和 Dirichlet 空间上 Topelitz 算子与小 Hankel 算子的矩阵表示. 它们表明调和 Dirichlet 空间上 Topelitz 算子与小 Hankel 算子与 Dirichlet 空间上 Topelitz 算子与小 Hankel 算子有着密切的联系. 这一思想来自于文 [17].

引理 2.2 [9] 设 $\phi \in \mathcal{M}$, 则在 $\mathcal{D} \oplus \mathbb{C} \oplus \mathcal{D}$ 上,

$$\tilde{U}\tilde{T}_\phi\tilde{U}^* = \begin{pmatrix} T_\phi & P_\phi & \varGamma_{\hat{\phi}} \\ 1 \otimes \phi & 1 \otimes \phi & 1 \otimes \hat{\phi} \\ \varGamma_\phi & P_{\hat{\phi}} & T_{\hat{\phi}} \end{pmatrix}.$$

引理 2.3 设 $\phi \in \mathcal{M}$. 则在 $\mathcal{D} \oplus \mathbb{C} \oplus \mathcal{D}$ 上,

$$\tilde{U}\tilde{\varGamma}_\phi\tilde{U}^* = \begin{pmatrix} \varGamma_\phi & P_{\hat{\phi}} & T_{\hat{\phi}} \\ 1 \otimes \phi & 1 \otimes \hat{\phi} & 1 \otimes \hat{\phi} \\ T_\phi & P_\phi & \varGamma_{\hat{\phi}} \end{pmatrix}.$$

证 设 $(f_1,\ a,\ f_2)\in\mathcal{D}\oplus\mathbb{C}\oplus\mathcal{D}$, 直接计算可得

$$\begin{aligned}
\widetilde{\varGamma}_\phi f_1 &= Q(U(\phi f_1))=P(U(\phi f_1))+\langle\hat{\phi}\hat{f}_1,1\rangle+P_1(U(\phi f_1))\\
&= \varGamma_\phi f_1+(1\otimes\hat{\phi})\hat{f}_1+UPU(U(\phi f_1))=\varGamma_\phi f_1+(1\otimes\hat{\phi})\hat{f}_1+UP(\phi f_1)\\
&= \varGamma_\phi f_1+(1\otimes\phi)f_1+UT_\phi f_1,\\
\widetilde{\varGamma}_\phi a &= Q(U(\phi a))=P(\hat{\phi}a)+\langle\hat{\phi}a,1\rangle+P_1(\hat{\phi}a)=P_{\hat{\phi}}(a)+(1\otimes\hat{\phi})(a)+UP_\phi(a),\\
\widetilde{\varGamma}_\phi\hat{f}_2 &= Q(U(\phi\hat{f}_2))=P(\hat{\phi}f_2)+\langle\hat{\phi}f_2,1\rangle+P_1(\hat{\phi}f_2)\\
&= P(\hat{\phi}f_2)+\langle\hat{\phi}f_2,1\rangle+UPU(\hat{\phi}f_2)=T_{\hat{\phi}}f_2+(1\otimes\hat{\phi})f_2+U\varGamma_{\hat{\phi}}f_2.
\end{aligned}$$

因此

$$\begin{aligned}
\widetilde{U}\widetilde{\varGamma}_\phi\widetilde{U}^*\begin{pmatrix}f_1\\ a\\ f_2\end{pmatrix} &= \widetilde{U}\widetilde{\varGamma}_\phi\begin{pmatrix}f_1\\ a\\ \hat{f}_2\end{pmatrix}=\widetilde{U}\begin{pmatrix}\varGamma_\phi f_1+P_{\hat{\phi}}(a)+T_{\hat{\phi}}f_2\\ (1\otimes\phi)f_1+(1\otimes\hat{\phi})(a)+(1\otimes\hat{\phi})f_2\\ UT_\phi f_1+UP_\phi(a)+U\varGamma_{\hat{\phi}}f_2\end{pmatrix}\\
&= \begin{pmatrix}\varGamma_\phi f_1+P_{\hat{\phi}}(a)+T_{\hat{\phi}}f_2\\ (1\otimes\phi)f_1+(1\otimes\hat{\phi})(a)+(1\otimes\hat{\phi})f_2\\ T_\phi f_1+P_\phi(a)+\varGamma_{\hat{\phi}}f_2\end{pmatrix}\\
&= \begin{pmatrix}\varGamma_\phi & P_{\hat{\phi}} & T_{\hat{\phi}}\\ 1\otimes\phi & 1\otimes\hat{\phi} & 1\otimes\hat{\phi}\\ T_\phi & P_\phi & \varGamma_{\hat{\phi}}\end{pmatrix}\begin{pmatrix}f_1\\ a\\ f_2\end{pmatrix}.
\end{aligned}$$

证毕.

为了方便后面的应用, 下面给出 $\mathcal{D}$ 上 Toeplitz 算子与 Hankel 算子及其它算子在标准正交基 $\{e_n=\frac{z^n}{\sqrt{n}}\}_{n=1}^\infty$ 下的系数.

引理 2.4 设 $\varphi(z)=\sum\limits_{k<0}a_k\bar{z}^{-k}+\sum\limits_{k\geq 0}a_kz^k$, $\psi(z)=\sum\limits_{k<0}b_k\bar{z}^{-k}+\sum\limits_{k\geq 0}b_kz^k\in\mathcal{M}$, 则

1° $\langle T_\varphi e_n,e_m\rangle=\frac{\sqrt{m}}{\sqrt{n}}a_{m-n}$, $\langle\varGamma_\varphi e_n,e_m\rangle=\frac{\sqrt{m}}{\sqrt{n}}a_{-(m+n)}$;

2° $\langle T_\varphi\varGamma_\psi e_n,e_m\rangle=\sum\limits_{k=1}^\infty\frac{\sqrt{m}}{\sqrt{n}}a_{m-k}b_{-n-k}$, $\langle\varGamma_\varphi T_\psi e_n,e_m\rangle=\sum\limits_{k=1}^\infty\frac{\sqrt{m}}{\sqrt{n}}a_{-m-k}b_{-n+k}$;

3° $\langle\varGamma_\varphi\varGamma_\psi e_n,e_m\rangle=\sum\limits_{k=1}^\infty\frac{\sqrt{m}}{\sqrt{n}}a_{-m-k}b_{-n-k}$, $\langle T_\varphi T_\psi e_n,e_m\rangle=\sum\limits_{k=1}^\infty\frac{\sqrt{m}}{\sqrt{n}}a_{m-k}b_{-n+k}$;

4° $(1\otimes\varphi)e_n=\frac{a_{-n}}{\sqrt{n}(n+1)}$, $\langle P_\varphi(1\otimes\psi)e_n,e_m\rangle=\frac{\sqrt{m}\,a_mb_{-n}}{\sqrt{n}(1+n)}$;

5° $\langle T_\varphi P_\psi 1,e_m\rangle=\sqrt{m}\sum\limits_{k=1}^\infty a_{m-k}b_k$, $\langle P_\varphi(1\otimes\psi)1,e_m\rangle=\sqrt{m}\,a_mb_0$,

$$\langle\varGamma_\varphi P_\psi 1,e_m\rangle=\sqrt{m}\sum_{k=1}^\infty a_{-m-k}b_k.$$

证 直接计算可知 $\langle T_\varphi e_n,e_m\rangle=\frac{\sqrt{m}}{\sqrt{n}}a_{m-n}$, $\langle\varGamma_\varphi e_n,e_m\rangle=\frac{\sqrt{m}}{\sqrt{n}}a_{-(m+n)}$. 因此

$$T_\varphi e_n=\sum_{k=1}^\infty\frac{\sqrt{k}}{\sqrt{n}}a_{k-n}e_k,\varGamma_\varphi e_n=\sum_{k=1}^\infty\frac{\sqrt{k}}{\sqrt{n}}a_{-(k+n)}e_k.$$

$$\begin{aligned}
\langle T_\varphi\Gamma_\psi e_n, e_m\rangle &= \langle T_\varphi(\sum_{k=1}^{\infty}\frac{\sqrt{k}}{\sqrt{n}}b_{-n-k}e_k), e_m\rangle = \sum_{k=1}^{\infty}\frac{\sqrt{k}}{\sqrt{n}}b_{-n-k}\langle T_\varphi e_k, e_m\rangle \\
&= \sum_{k=1}^{\infty}\frac{\sqrt{m}}{\sqrt{n}}a_{m-k}b_{-n-k}, \\
\langle \Gamma_\varphi T_\psi e_n, e_m\rangle &= \langle \Gamma_\varphi(\sum_{k=1}^{\infty}\frac{\sqrt{k}}{\sqrt{n}}b_{-n+k}e_k), e_m\rangle = \sum_{k=1}^{\infty}\frac{\sqrt{k}}{\sqrt{n}}b_{-n+k}\langle \Gamma_\varphi e_k, e_m\rangle \\
&= \sum_{k=1}^{\infty}\frac{\sqrt{m}}{\sqrt{n}}a_{-m-k}b_{-n+k}, \\
\langle \Gamma_\varphi\Gamma_\psi e_n, e_m\rangle &= \langle \Gamma_\varphi(\sum_{k=1}^{\infty}\frac{\sqrt{k}}{\sqrt{n}}b_{-n-k}e_k), e_m\rangle = \sum_{k=1}^{\infty}\frac{\sqrt{k}}{\sqrt{n}}b_{-n-k}\langle \Gamma_\varphi e_k, e_m\rangle \\
&= \sum_{k=1}^{\infty}\frac{\sqrt{m}}{\sqrt{n}}a_{-m-k}b_{-n-k}, \\
\langle T_\varphi T_\psi e_n, e_m\rangle &= \langle T_\varphi(\sum_{k=1}^{\infty}\frac{\sqrt{k}}{\sqrt{n}}b_{-n+k}e_k), e_m\rangle = \sum_{k=1}^{\infty}\frac{\sqrt{k}}{\sqrt{n}}b_{-n+k}\langle T_\varphi e_k, e_m\rangle \\
&= \sum_{k=1}^{\infty}\frac{\sqrt{m}}{\sqrt{n}}a_{m-k}b_{-n+k}.
\end{aligned}$$

同样计算可得

$$\begin{aligned}
(1\otimes\varphi)e_n &= \int_{\mathbb{D}}\varphi e_n dA = \int_{\mathbb{D}}\big(\sum_{k<0}a_k\bar{z}^{-k}+\sum_{k\geq 0}a_k z^k\big)\frac{z^n}{\sqrt{n}}dA(z) = \frac{a_{-n}}{\sqrt{n}(n+1)}, \\
\langle P_\varphi(1\otimes\psi)e_n, e_m\rangle &= (1\otimes\psi)e_n\langle P(\varphi), e_m\rangle = \frac{b_{-n}}{\sqrt{n}(1+n)}\langle\sum_{k>0}a_k z^k, e_m\rangle = \frac{\sqrt{m}\,a_m b_{-n}}{\sqrt{n}(1+n)}, \\
\langle T_\varphi P_\psi 1, e_m\rangle &= \langle T_\varphi P(\psi), e_m\rangle = \sum_{k>0}b_k\sqrt{k}\,\langle T_\varphi e_k, e_m\rangle = \sqrt{m}\sum_{k=1}^{\infty}a_{m-k}b_k, \\
\langle P_\varphi(1\otimes\psi)1, e_m\rangle &= ((1\otimes\psi)1)\langle P(\varphi), e_m\rangle = b_0\,\langle\sum_{k>0}a_k z^k\ ,\ e_m\rangle = \sqrt{m}\,a_m b_0, \\
\langle \Gamma_\varphi P_\psi 1, e_m\rangle &= \langle \Gamma_\varphi P(\psi), e_m\rangle = \sum_{k>0}b_k\sqrt{k}\,\langle \Gamma_\varphi e_k\ ,\ e_m\rangle = \sqrt{m}\sum_{k=1}^{\infty}a_{-m-k}b_k.
\end{aligned}$$

下面我们给出本文主要定理 1.1 的证明.

证 充分性的证明.

当 φ 是常数时, 则 $\widetilde{T}_\varphi\widetilde{\Gamma}_\psi = \widetilde{\Gamma}_\psi\widetilde{T}_\varphi$ 显然成立.

当 φ 不为常数, $\psi = \alpha\varphi+\beta$, 且当 $\alpha\neq 0$ 或 $\beta\neq 0$ 时, $\varphi=\hat{\varphi}$, 则 $\forall\, f\in\mathcal{D}_h$,

$$\begin{aligned}
\widetilde{T}_\varphi\widetilde{\Gamma}_\psi f &= \widetilde{T}_\varphi\widetilde{\Gamma}_{\alpha\varphi}f+\widetilde{T}_\varphi\widetilde{\Gamma}_\beta f = \widetilde{T}_\varphi Q(U(\alpha\varphi f))+\widetilde{T}_\varphi Q(U(\beta f)) \\
&= \alpha\widetilde{T}_\varphi UQ(\varphi f)+\beta\widetilde{T}_\varphi Uf = \alpha Q(\varphi UQ(\varphi f))+\beta Q(\varphi Uf) = \alpha\widetilde{\Gamma}_{\hat{\varphi}}\widetilde{T}_\varphi f+\beta\widetilde{\Gamma}_{\hat{\varphi}}f, \\
\widetilde{\Gamma}_\psi\widetilde{T}_\varphi f &= \widetilde{\Gamma}_{\alpha\varphi}\widetilde{T}_\varphi f+\widetilde{\Gamma}_\beta\widetilde{T}_\varphi f = Q(U(\alpha\varphi\widetilde{T}_\varphi f))+Q(\beta UQ(\varphi f)) \\
&= \alpha Q(U(\varphi\widetilde{T}_\varphi f))+Q(\beta QU(\varphi f)) = \alpha\widetilde{\Gamma}_\varphi\widetilde{T}_\varphi f+\beta\widetilde{\Gamma}_\varphi f.
\end{aligned}$$

在上面两式中用到引理 2.1.

比较上面两式可知, 当 $\alpha \neq 0$ 或 $\beta \neq 0$, 且 $\varphi = \hat{\varphi}$ 时, $\widetilde{T}_\varphi \widetilde{\Gamma}_\psi = \widetilde{\Gamma}_\psi \widetilde{T}_\varphi$.

必要性的证明.

若 $\widetilde{T}_\varphi \widetilde{\Gamma}_\psi = \widetilde{\Gamma}_\psi \widetilde{T}_\varphi$, 则 $\widetilde{U}\widetilde{T}_\varphi \widetilde{U}^* \widetilde{U}\widetilde{\Gamma}_\psi \widetilde{U}^* = \widetilde{U}\widetilde{\Gamma}_\psi \widetilde{U}^* \widetilde{U}\widetilde{T}_\varphi \widetilde{U}^*$, 由引理 2.2 与引理 2.3 得

$$\begin{aligned} &\begin{pmatrix} T_\varphi & P_\varphi & \Gamma_{\hat{\varphi}} \\ 1\otimes\varphi & 1\otimes\varphi & 1\otimes\hat{\varphi} \\ \Gamma_\varphi & P_{\hat{\varphi}} & T_{\hat{\varphi}} \end{pmatrix} \begin{pmatrix} \Gamma_\psi & P_{\hat{\psi}} & T_{\hat{\psi}} \\ 1\otimes\psi & 1\otimes\hat{\psi} & 1\otimes\hat{\psi} \\ T_\psi & P_\psi & \Gamma_{\hat{\psi}} \end{pmatrix} \\ = &\begin{pmatrix} \Gamma_\psi & P_{\hat{\psi}} & T_{\hat{\psi}} \\ 1\otimes\psi & 1\otimes\hat{\psi} & 1\otimes\hat{\psi} \\ T_\psi & P_\psi & \Gamma_{\hat{\psi}} \end{pmatrix} \begin{pmatrix} T_\varphi & P_\varphi & \Gamma_{\hat{\varphi}} \\ 1\otimes\varphi & 1\otimes\varphi & 1\otimes\hat{\varphi} \\ \Gamma_\varphi & P_{\hat{\varphi}} & T_{\hat{\varphi}} \end{pmatrix}. \end{aligned}$$

因此

$$T_\varphi\Gamma_\psi + P_\varphi(1\otimes\psi) + \Gamma_{\hat{\varphi}}T_\psi = \Gamma_\psi T_\varphi + P_{\hat{\psi}}(1\otimes\varphi) + T_{\hat{\psi}}\Gamma_\varphi, \tag{2.1}$$

$$T_\varphi T_{\hat{\psi}} + P_\varphi(1\otimes\hat{\psi}) + \Gamma_{\hat{\varphi}}\Gamma_{\hat{\psi}} = \Gamma_\psi \Gamma_{\hat{\varphi}} + P_{\hat{\psi}}(1\otimes\hat{\varphi}) + T_{\hat{\psi}}T_{\hat{\varphi}}, \tag{2.2}$$

$$\Gamma_\varphi\Gamma_\psi + P_{\hat{\varphi}}(1\otimes\psi) + T_{\hat{\varphi}}T_\psi = T_\psi T_\varphi + P_\psi(1\otimes\varphi) + \Gamma_{\hat{\psi}}\Gamma_\varphi, \tag{2.3}$$

$$\Gamma_\varphi T_{\hat{\psi}} + P_{\hat{\varphi}}(1\otimes\hat{\psi}) + T_{\hat{\varphi}}\Gamma_{\hat{\psi}} = T_\psi \Gamma_{\hat{\varphi}} + P_\psi(1\otimes\hat{\varphi}) + \Gamma_{\hat{\psi}}T_{\hat{\varphi}}, \tag{2.4}$$

$$T_\varphi P_{\hat{\psi}} + P_\varphi(1\otimes\hat{\psi}) + \Gamma_{\hat{\varphi}}P_\psi = \Gamma_\psi P_\varphi + P_{\hat{\psi}}(1\otimes\varphi) + T_{\hat{\psi}}P_{\hat{\varphi}}, \tag{2.5}$$

$$\Gamma_\varphi P_{\hat{\psi}} + P_{\hat{\varphi}}(1\otimes\hat{\psi}) + T_{\hat{\varphi}}P_\psi = T_\psi P_\varphi + P_\psi(1\otimes\varphi) + \Gamma_{\hat{\psi}}P_{\hat{\varphi}}, \tag{2.6}$$

$$(1\otimes\varphi)P_{\hat{\psi}} + (1\otimes\varphi)(1\otimes\hat{\psi}) + (1\otimes\hat{\varphi})P_\psi = (1\otimes\psi)P_\varphi + (1\otimes\hat{\psi})(1\otimes\varphi) + (1\otimes\hat{\psi})P_{\hat{\varphi}}. \tag{2.7}$$

设 $\varphi(z) = \sum\limits_{k<0} a_k \bar{z}^{-k} + \sum\limits_{k\geq 0} a_k z^k$, $\psi(z) = \sum\limits_{k<0} b_k \bar{z}^{-k} + \sum\limits_{k\geq 0} b_k z^k$, 则

$$\hat{\varphi}(z) = \sum_{k<0} a_{-k}\bar{z}^{-k} + \sum_{k\geq 0} a_{-k} z^k, \quad \hat{\psi}(z) = \sum_{k<0} b_{-k}\bar{z}^{-k} + \sum_{k\geq 0} b_{-k} z^k.$$

由引理 2.4 计算得

$$\begin{aligned} & \langle (T_\varphi\Gamma_\psi + P_\varphi(1\otimes\psi) + \Gamma_{\hat{\varphi}}T_\psi)e_n, e_m\rangle \\ = & \sum_{k=1}^{\infty} \frac{\sqrt{m}}{\sqrt{n}} a_{m-k}b_{-n-k} + \frac{\sqrt{m}}{\sqrt{n}}\frac{a_m b_{-n}}{1+n} + \sum_{k=1}^{\infty}\frac{\sqrt{m}}{\sqrt{n}} a_{m+k}b_{k-n}, \\ & \langle (\Gamma_\psi T_\varphi + P_{\hat{\psi}}(1\otimes\varphi) + T_{\hat{\psi}}\Gamma_\varphi)e_n, e_m\rangle \\ = & \sum_{k=1}^{\infty} \frac{\sqrt{m}}{\sqrt{n}} a_{-n+k}b_{-m-k} + \frac{\sqrt{m}}{\sqrt{n}}\frac{a_{-n} b_{-m}}{1+n} + \sum_{k=1}^{\infty}\frac{\sqrt{m}}{\sqrt{n}} a_{-n-k}b_{-m+k}. \end{aligned}$$

由 (2.1) 式得

$$\sum_{k=-\infty}^{\infty} a_{m+k}b_{-n+k} + (\frac{1}{1+n}-1)a_m b_{-n} = \sum_{k=-\infty}^{\infty} a_{-n+k}b_{-m-k} + (\frac{1}{1+n}-1)a_{-n}b_{-m}. \tag{2.1$'$}$$

同样计算 (2.2), (2.3), (2.4) 式得

$$\sum_{k=-\infty}^{\infty} a_{m+k}b_{n+k} + (\frac{1}{1+n}-1)a_m b_n = \sum_{k=-\infty}^{\infty} a_{n+k}b_{-m-k} + (\frac{1}{1+n}-1)a_n b_{-m}, \tag{2.2$'$}$$

$$\sum_{k=-\infty}^{\infty} a_{-m+k}b_{-n+k} + (\frac{1}{1+n}-1)a_{-m} b_{-n} = \sum_{k=-\infty}^{\infty} a_{-n+k}b_{m-k} + (\frac{1}{1+n}-1)a_{-n} b_{m}, \tag{2.3$'$}$$

$$\sum_{k=-\infty}^{\infty} a_{-m+k}b_{n+k} + (\frac{1}{1+n}-1)a_{-m} b_{n} = \sum_{k=-\infty}^{\infty} a_{n+k}b_{m-k} + (\frac{1}{1+n}-1)a_{n} b_{m}. \tag{2.4$'$}$$

由引理 2.4 得

$$\langle (T_\varphi P_{\hat{\psi}} + P_\varphi(1\otimes\hat{\psi}) + \Gamma_{\hat{\varphi}}P_\psi)1\,, e_m\rangle = \sqrt{m}\sum_{k=1}^{\infty} a_{m-k}b_{-k} + \sqrt{m}\, a_m b_0 + \sqrt{m}\sum_{k=1}^{\infty} a_{m+k}b_k,$$

$$\langle (\Gamma_\psi P_\varphi + P_{\hat{\psi}}(1\otimes\varphi) + T_{\hat{\psi}}P_{\hat{\varphi}})1\,, e_m\rangle = \sqrt{m}\sum_{k=1}^{\infty} a_k b_{-m-k} + \sqrt{m}\, a_0\, b_{-m} + \sqrt{m}\,\sum_{k=1}^{\infty} a_{-k}b_{-m+k}.$$

由 (2.5) 式得

$$\sum_{k=-\infty}^{\infty} a_{m+k}b_k = \sum_{k=-\infty}^{\infty} a_k b_{-m-k}. \tag{2.5$'$}$$

同样计算 (2.6) 式可得

$$\sum_{k=-\infty}^{\infty} a_{-m+k}b_k = \sum_{k=-\infty}^{\infty} a_k b_{m-k}. \tag{2.6$'$}$$

因为

$$\begin{aligned}
\langle (1\otimes\varphi)P_{\hat{\psi}}1, 1\rangle &= \langle (1\otimes\varphi)P(\hat{\psi}), 1\rangle = \langle (1\otimes\varphi)\sum_{k>0} b_{-k}z^k, 1\rangle \\
&= \sum_{k>0} b_{-k}\sqrt{k}\langle (1\otimes\varphi)e_k, 1\rangle = \sum_{k=1}^{\infty}\frac{a_{-k}b_{-k}}{1+k}, \\
\langle (1\otimes\varphi)(1\otimes\hat{\psi})1, 1\rangle &= a_0 b_0, \\
\langle (1\otimes\hat{\varphi})P_\psi 1, 1\rangle &= \langle (1\otimes\hat{\varphi})P(\psi), 1\rangle = \langle (1\otimes\hat{\varphi})\sum_{k>0} b_k z^k, 1\rangle \\
&= \sum_{k>0} b_k\sqrt{k}\,\langle (1\otimes\hat{\varphi})e_k, 1\rangle = \sum_{k=1}^{\infty}\frac{a_k b_k}{1+k},
\end{aligned}$$

所以

$$\langle ((1\otimes\varphi)P_{\hat{\psi}} + (1\otimes\varphi)(1\otimes\hat{\psi}) + (1\otimes\hat{\varphi})P_\psi)1, 1\rangle = \sum_{k=1}^{\infty}\frac{a_{-k}b_{-k}}{1+k} + a_0 b_0 + \sum_{k=1}^{\infty}\frac{a_k b_k}{1+k},$$

$$\langle ((1\otimes\psi)P_{\varphi} + (1\otimes\hat{\psi})(1\otimes\varphi) + (1\otimes\hat{\psi})P_{\hat{\varphi}})1, 1\rangle = \sum_{k=1}^{\infty}\frac{a_k b_{-k}}{1+k} + a_0 b_0 + \sum_{k=1}^{\infty}\frac{a_{-k} b_k}{1+k}.$$

由 (2.7) 式得

$$\sum_{k=1}^{\infty}\frac{a_{-k}b_{-k}}{1+k}+a_0b_0+\sum_{k=1}^{\infty}\frac{a_kb_k}{1+k}=\sum_{k=1}^{\infty}\frac{a_kb_{-k}}{1+k}+a_0b_0+\sum_{k=1}^{\infty}\frac{a_{-k}b_k}{1+k}. \tag{2.7$'$}$$

在 $(2.1')$ 与 $(2.3')$ 式中令 $-n+k=j$, 则 $k=n+j$,

$$\sum_{j=-\infty}^{\infty}a_{m+n+j}b_j+(\frac{1}{1+n}-1)a_mb_{-n}=\sum_{j=-\infty}^{\infty}a_jb_{-m-n-j}+(\frac{1}{1+n}-1)a_{-n}b_{-m}, \tag{2.1$''$}$$

$$\sum_{j=-\infty}^{\infty}a_{n-m+j}b_j+(\frac{1}{1+n}-1)a_{-m}b_{-n}=\sum_{j=-\infty}^{\infty}a_jb_{m-n-j}+(\frac{1}{1+n}-1)a_{-n}b_m. \tag{2.3$''$}$$

在 $(2.2')$ 与 $(2.4')$ 中令 $n+k=j$, 则 $k=-n+j$,

$$\sum_{j=-\infty}^{\infty}a_{m-n+j}b_j+(\frac{1}{1+n}-1)a_mb_n=\sum_{j=-\infty}^{\infty}a_jb_{n-m-j}+(\frac{1}{1+n}-1)a_nb_{-m}, \tag{2.2$''$}$$

$$\sum_{j=-\infty}^{\infty}a_{-m-n+j}b_j+(\frac{1}{1+n}-1)a_{-m}b_n=\sum_{j=-\infty}^{\infty}a_jb_{m+n-j}+(\frac{1}{1+n}-1)a_nb_m. \tag{2.4$''$}$$

对任意正整数 m,n, 由 $(2.1'')$ 和 $(2.5')$ 式结合可得

$$a_mb_{-n}=a_{-n}b_{-m}. \tag{2.8}$$

由 $(2.3'')$ 和 $(2.5'),(2.6')$ 式结合可得

$$a_{-m}b_{-n}=a_{-n}b_m\ \ (m\neq n). \tag{2.9}$$

由 $(2.2'')$ 和 $(2.5'),(2.6')$ 式结合可得

$$a_mb_n=a_nb_{-m}\ \ (m\neq n). \tag{2.10}$$

由 $(2.4'')$ 和 $(2.6')$ 式结合可得

$$a_{-m}b_n=a_nb_m. \tag{2.11}$$

在 $(2.2''),(2.3'')$ 式中令 $m=n$, 则

$$\sum_{j=-\infty}^{\infty}a_jb_j+(\frac{1}{n+1}-1)a_{-n}b_{-n}=\sum_{j=-\infty}^{\infty}a_jb_{-j}+(\frac{1}{n+1}-1)a_{-n}b_n,$$
$$\sum_{j=-\infty}^{\infty}a_jb_j+(\frac{1}{n+1}-1)a_nb_n=\sum_{j=-\infty}^{\infty}a_jb_{-j}+(\frac{1}{n+1}-1)a_nb_{-n}.$$

即有

$$\sum_{j=-\infty}^{\infty}a_jb_j-\sum_{j=-\infty}^{\infty}a_jb_{-j}=(\frac{1}{1+n}-1)(a_{-n}b_n-a_{-n}b_{-n}),$$
$$\sum_{j=-\infty}^{\infty}a_jb_j-\sum_{j=-\infty}^{\infty}a_jb_{-j}=(\frac{1}{1+n}-1)(a_nb_{-n}-a_nb_n).$$

对比以上两式可得 $a_{-n}b_n - a_{-n}b_{-n} = a_n b_{-n} - a_n b_n$, 即

$$(a_{-n} + a_n)(b_n - b_{-n}) = 0. \tag{2.12}$$

从而 $a_{-n} = -a_n$ 或 $b_{-n} = b_n$.

设 φ 不是常数, 则存在正整数 m_0 使得 $a_{m_0} \neq 0$ 或 $a_{-m_0} \neq 0$. 由对称性, 不妨设 $a_{m_0} \neq 0$.

对任意正整数 n, 由 (2.8) 式得

$$b_{-n} = \frac{b_{-m_0}}{a_{m_0}} a_{-n}. \tag{2.13}$$

由 (2.10) 式得

$$b_n = \frac{b_{-m_0}}{a_{m_0}} a_n (n \neq m_0). \tag{2.14}$$

若 $b_{-m_0} \neq 0$, 则由 (2.13) 式得 $a_{-m_0} = a_{m_0} \neq 0$, 由 (2.12) 式得 $b_{m_0} = b_{-m_0} = \frac{b_{-m_0}}{a_{m_0}} a_{m_0}$. 令 $\alpha_1 = \frac{b_{-m_0}}{a_{m_0}}$, 则存在常数 β_1, 使得 $\psi = \alpha_1 \varphi + \beta_1$.

若 $b_{-m_0} = 0$, 则由 (2.13), (2.14) 式得 $b_{-n} = 0$, $b_n = 0 (n \neq m_0)$. 由 (2.7′) 式得

$$a_0 b_0 + \frac{a_{m_0} b_{m_0}}{1 + m_0} = a_0 b_0 + \frac{a_{-m_0} b_{m_0}}{1 + m_0}.$$

因此有 $(a_{m_0} - a_{-m_0}) b_{m_0} = 0$. 因为 $b_{-m_0} = 0$, 由 (2.12) 式得 $(a_{-m_0} + a_{m_0}) b_{m_0} = 0$. 若 $b_{m_0} \neq 0$, 则 $a_{-m_0} + a_{m_0} = 0$, 且 $a_{m_0} - a_{-m_0} = 0$, 因此有 $a_{m_0} = 0$, 矛盾. 所以 $b_{m_0} = 0$. 因此 ψ 是一个常数. 设 $\psi = \beta_2$, 令 $\alpha_2 = 0$, 则有 $\psi = \alpha_2 \varphi + \beta_2$.

以下设 $\psi = \alpha\varphi + \beta$, 则对于任意正整数 n, $b_n = \alpha a_n$, $b_{-n} = \alpha a_{-n}$.

当 $\alpha \neq 0$ 时, 代入 (2.11) 式得 $a_{-m} a_n = a_n a_m$. 因为 $a_{m_0} \neq 0$, 所以对任意正整数 n, 有 $a_n = a_{-n}$. 因此 $\varphi = U\varphi$.

当 $\alpha = 0$, $\beta \neq 0$ 时, 由 $\widetilde{T}_\varphi \widetilde{\varGamma}_\psi = \widetilde{\varGamma}_\psi \widetilde{T}_\varphi$ 得 $\widetilde{T}_\varphi \widetilde{\varGamma}_\psi(1) = \widetilde{\varGamma}_\psi \widetilde{T}_\varphi(1)$. 直接计算得 $\varphi = U\varphi$. 证毕.

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COMMUTATIVITY OF TOEPLITZ OPERATOR AND SMALL HANKEL OPERATOR ON HARMONIC DIRICHLET SPACE

FENG Li-xia [1,2], MA Wen-fang [3], ZHAO Lian-kuo [2]

(*1. Research Center for History of Mathematics and Science, Northwest University, Xi'an 710127, China*)

(*2.School of Mathematics and Computer Sciences, Shanxi Normal University, Linfen 041004, China*)

(*3.First Middle School of Shuozhou City, Shuozhou 036000, China*)

Abstract: In this paper, we study the commutativity of Toeplitz operator and small Hankel operator on harmonic Dirichlet space. By the matrix representation of Toeplitz operator and small Hankel operator, we obtain the necessity and sufficient condition for Toeplitz operator and small Hankel operator on harmonic Dirichlet space to be commutativity, which extends the corresponding results on Dirichlet space.

Keywords: harmonic Dirichlet space; Toeplitz operator; small Hankel operator; commutativity

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