第26卷第1期 数 学 研 究 与 评 论 Vol.26, No.1
2006年2月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION Feb., 2006

Article ID: 1000-341X(2006)01-0027-06 **Document code**: A

Generalized IP-Injective Rings

MAO Li-xin[1,2], *TONG Wen-ting*[2]

(1. Dept. of Basic Courses, Nanjing Institute of Technology, Jiangsu 210013, China;
2. Dept. of Math., Nanjing University, Jiangsu 210093, China)
(E-mail: maolx2@hotmail.com)

Abstract: For a ring R, let $ip(R_R) = \{a \in R$: every right R-homomorphism f from any right ideal of R into R with $\mathrm{Im} f = aR$ can extend to $R\}$. It is known that R is right IP-injective if and only if $R = ip(R_R)$ and R is right simple-injective if and only if $\{a \in R : aR \text{ is simple}\} \subseteq ip(R_R)$. In this note, we introduce the concept of right S-IP-injective rings, i.e., the ring R with $S \subseteq ip(R_R)$, where S is a subset of R. Some properties of this kind of rings are obtained.

Key words: S-IP-injective ring; simple-injective ring; $C2$-ring.
MSC(2000): 16D10, 16D50
CLC number: O153.3

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary. S always denotes a subset of a ring R. As usual, we use $J(R)$, $Z({}_RR)$, $Z(R_R)$, $\mathrm{Soc}({}_RR)$ and $\mathrm{Soc}(R_R)$ to indicate the Jacobson radical, the left singular ideal, right singular ideal, and the left socle and right socle of the ring R, respectively. The left and right annihilators of a subset X of R are denoted by $l(X)$ and $r(X)$, respectively.

Recall that a ring R is called right P-injective[1] if every right R-homomorphism from any principal right ideal of R into R is given by left multiplication by an element of R. P-injective rings and their generalizations have been studied in many papers such as [1–4]. Recently, Chen and Ding[4] define the concept of IP-injective rings, i.e., a ring R is said to be right IP-injective if every right R-homomorphism from any right ideal of R into R with principal image is given by left multiplication by an element of R. It is proved in [4] that a ring R is right IP-injective if and only if R is right P-injective and right GIN (i.e., $l(I \cap K) = l(I) + l(K)$ for each pair of right ideals I and K with I principal). In this paper, we introduce a generalization of IP-injective rings, which are called right S-IP-injective rings, i.e., a ring R satisfying the condition that every right R-homomorphism $I \to R$ with $\mathrm{Im} f = aR$, $a \in S$, where I is a right ideal of R and S is a subset of R, is given by left multiplication by an element of R. We show the following results: (1) If R is a right $GC2$-ring such that $l(aR \cap K) = l(a) + l(K)$ for each pair of right ideals K and aR of R with $r(a) = 0$, then R is a right S-IP-injective ring with $S = \{a \in R : r(a) = 0\}$. (2) If

Received date: 2003-07-16
Foundation item: the Specialized Research Fund for the Doctoral Program of Higher Education of China (20020284009, 20030284033), the Postdoctoral Research Fund of China (2005037713), Jiangsu Planned Projects for Postdoctoral Research Fund (0203003403).

R is right $J(R)$-IP-injective, and left Kasch, then R is right JP-injective with $Z(R_R) = J(R)$. (3) If R is a right $\mathcal{I}(R) \cup J(R)$-IP-injective ring where $\mathcal{I}(R) = \{$all idempotents of the ring $R\}$, then R is a right simple-injective ring. (4) Every semiregular right $\mathcal{I}(R) \cup J(R)$-IP-injective ring is right IP-injective. (5) A ring R is a quasi-Frobenius ring if and only if R is a left Noetherian, right ACS and right $\mathcal{I}(R) \cup J(R)$-IP-injective ring.

General background material can be found in [5].

2. Main results

Notation 2.1 For a ring R , let $ip(R_R) = \{a \in R :$ for any right ideal I of R, every right R-homomorphism f: $I \to R$ with Im$f = aR$ can extend to R, or equivalently, f is given by left multiplication by an element of $R\}$.

Definition 2.2 *A ring R is called right S-IP-injective if $S \subseteq ip(R_R)$, where S is a subset of R.*

Remark 2.3 It is easy to see that a ring R is right IP-injective if and only if R is right R-IP-injective. Recall that a ring R is called right simple-injective[3] if every right R-homomorphism f: $I \to R$ with simple Imf is given by left multiplication by an element of R. It is obvious that a ring R is right simple-injective if and only if R is right S-IP-injective with $S = \{a \in R : aR$ is simple$\}$. So it seems reasonable to raise the question of how the structure of a ring R is determined by properties of certain S contained in $ip(R_R)$.

Proposition 2.4 *Let $a \in ip(R_R)$. Then $R = l(a) + l(I)$ where I is any right ideal of R such that $aR \bigcap I = 0$.*

Proof Assume I is a right ideal such that $aR \bigcap I = 0$. Let $\pi : aR \oplus I \longrightarrow aR$ denote the canonical projection. It is clear that Im$\pi = aR$. By hypothesis, it follows that π is given by left multiplication by an element c of R. This means that $a = \pi(a + b) = c(a + b)$ for any $b \in I$. So $1 - c \in l(a)$, $c \in l(b)$. Since $1 = 1 - c + c \in l(a) + l(I)$, then $R = l(a) + l(I)$. □

Lemma 2.5 *Let R be a ring. Then the following are equivalent:*

(1) Every right R-homomorphism $f : I \longrightarrow R_R$ where I is a right ideal of R with Im$f \cong R_R$ can extend to R;

(2) R is a right S-IP-injective ring with $S = \{a \in R : r(a) = 0\}$.

Proof (1) $\Rightarrow$ (2). Let $a \in R$ with $r(a) = 0$. Then $aR \cong R_R$. For every right R-homomorphism $f : I \longrightarrow R_R$ with Im$f = aR$, we have Im$f \cong R_R$. By (1), f can extend to R, and (2) holds.

(2) $\Rightarrow$ (1). Let $f : I \longrightarrow R_R$ be a right R-homomorphism with Im$f \cong R_R$. So Im$f = aR$ for some $a \in R$. Suppose σ: $aR \to R_R$ is the R-module isomorphism, then there exists $c \in R$ such that $ac = \sigma^{-1}(1)$. Note that $acR = aR =$ Imf and $r(ac) = 0$, thus $ac \in ip(R_R)$ by (2), so f can extend to R, as required. □

Recall that a ring R is called a right $C2$-ring if for any right ideal I with $I \cong K$ where K is a direct summand of R_R, I is a direct summand of R_R. Following [2], a ring R is called a right generalized $C2$-ring (or $GC2$-ring) if for any right ideal I with $I \cong R_R$, I is a direct summand of R_R.

Proposition 2.6 *If R is a right S-IP-injective ring with $S = \{a \in R : r(a) = 0\}$, then R is a right $GC2$-ring, and $R = l(I) + l(L)$ where I and L are right ideals of R such that $I \cong R_R$ and $I \bigcap L = 0$.*

Proof By Lemma 2.5, the first statement follows from [2, Proposition 2.2] and the second statement holds by a slight modification of the proof of Proposition 2.4. □

Next we consider the converse of Proposition 2.6.

Proposition 2.7 *If R is a right $GC2$-ring such that $l(aR \cap K) = l(a) + l(K)$ for each pair of right ideals K and aR of R with $r(a) = 0$, then R is a right S-IP-injective ring with $S = \{a \in R : r(a) = 0\}$.*

Proof First we suppose that $f : aR + I \to R_R$ is a right R-homomorphism such that both $f|_{aR} : aR \to R_R$ and $f|_I : I \to R_R$ are given by left multiplication by elements z_1 and z_2 of R, respectively, where I is a right ideal and $a \in R$ with $r(a) = 0$. Assume $x \in aR \cap I$, then $z_1x = z_2x$, and hence $z_1 - z_2 \in l(aR \cap I) = l(a) + l(I)$ by hypothesis. Thus $z_1 - z_2 = y_1 + y_2$ for some $y_1 \in l(a)$, $y_2 \in l(I)$. Now let $b_1 \in aR$, $b_2 \in I$, then $y_1b_1 = 0$, $y_2b_2 = 0$, and therefore $f(b_1 + b_2) = z_1b_1 + z_2b_2 = (z_1 - y_1)b_1 + (z_2 + y_2)b_2$. But $z_1 - y_1 = z_2 + y_2$, so $f(b_1 + b_2) = (z_1 - y_1)(b_1 + b_2)$. It follows that $f : aR + I \to R_R$ is given by left multiplication.

Then, we suppose that L is a right ideal of R and $f : L \to R_R$ is a right R-homomorphism with $\mathrm{Im} f = aR$ such that $r(a) = 0$. Let $a = f(b)$, for some $b \in L$, then $\mathrm{Im} f = aR = f(b)R$. It is easy to verify that $L = bR + \mathrm{Ker} f$. Since $r(a) = 0$, we claim $r(b) = 0$. Indeed, assume $x \in r(b)$, then $bx = 0$, $ax = f(b)x = f(bx) = f(0) = 0$. So $x \in r(a)$, and hence $x = 0$. It follows that $f|_{bR}$ is given by left multiplication by [2, Proposition 2.2] since R is a right $GC2$-ring. Clearly $f|_{Kerf}$ is also given by left multiplication by 0. Hence by earlier part of the proof, f is given by left multiplication. The proof is complete. □

It is obvious that right IP-injective rings are right S-IP-injective rings for any S. In general, the converse is not true. In fact, the ring $\mathbf{Z}$ of integers is $J(R)$-IP-injective, but not $GC2$, and hence it is not IP-injective by Proposition 2.6.

We know that right IP-injective rings satisfy the right GIN conditions by [4, Theorem 2.2]. Generally, for a right S-IP-injective ring, where S is a left ideal, we have the next corresponding proposition.

Proposition 2.8 *If a ring R is right S-IP-injective, where S is a left ideal of R, then $l(aR \cap K) = l(a) + l(K)$ for each pair of right ideals K and aR of R with $a \in S$.*

Proof It is clear that $l(aR \cap K) \supseteq l(a) + l(K)$. Conversely, let $t \in l(aR \cap K)$. Define a right R-homomorphism $\alpha : aR + K \to R_R$ via $ar + k \mapsto tar$ for $r \in R$, $k \in K$. Then it is easy to see that α is well-defined and $\mathrm{Im}\alpha = taR$. Since S is a left ideal and $a \in S$, then $ta \in S$. By hypothesis, α is given by left multiplication by an element c of R, so $tar = c(ar + k)$ for all $r \in R$ and all $k \in K$. Let $r = 1$, $k = 0$, then $t - c \in l(a)$, and let $r = 0$, then $c \in l(k)$. Thus $t = t - c + c \in l(a) + l(K)$. $\square$

Following [2], a ring R is said to be right JP-injective if every right R-module homomorphism $f : aR \to R_R$ where $a \in J(R)$ can extend to R. The next proposition give a relation between the $J(R)$-IP-injective rings and the JP-injective rings.

Recall that a ring R is called left Kasch if every simple left R-module can be embedded in ${}_RR$.

Proposition 2.9 *If a ring R is left Kasch right $J(R)$-IP-injective, then R is right JP-injective with $Z(R_R) = J(R)$.*

Proof Assume $a \in J(R)$ and $f : aR \to R_R$ is a right R-homomorphism. It follows that $f(a)\mathrm{Soc}({}_RR) = f(a\mathrm{Soc}({}_RR)) = f(0) = 0$, so $f(a) \in l(\mathrm{Soc}({}_RR))$. Since R is left Kasch, we have $J(R) = l(\mathrm{Soc}({}_RR))$. Therefore $f(a) \in J(R)$. Note that $\mathrm{Im}f = f(a)R$ and R is right $J(R)$-IP-injective, thus f can extend to R, and so R is right JP-injective. Consequently $J(R) \subseteq Z(R_R)$ by [2, Theorem 3.6]. In addition, by [6, Proposition 4.1], R is right $C2$ since R is left Kasch, and hence $Z(R_R) \subseteq J(R)$, which shows that $J(R) = Z(R_R)$ as desired. $\square$

From now on, $\mathcal{I}(R)$ always denotes the set $\{$all idempotents of a ring $R\}$.

Proposition 2.10 *Let R be a right $\mathcal{I}(R) \cup J(R)$-IP-injective ring. Then*

(1) R is a right $C2$-ring;

(2) If $a \in R$ such that aR is a simple right ideal or Ra is a simple left ideal, then for every right ideal I of R, every right R-homomorphism $f : I \to R$ with $\mathrm{Im}f = aR$ is given by left multiplication by an element of R. In particular, R is a right simple-injective ring.

Proof (1) Follows from [6, Proposition 4.4] since R is right $\mathcal{I}(R)$-IP-injective.

(2) Let f: $I \to R$ be a right R-homomorphism with $\mathrm{Im}f = aR$. If aR is simple and $(aR)^2 \neq 0$, then $aR = eR$ for some $e \in \mathcal{I}(R)$. Thus, by hypothesis, f is given by left multiplication by an element of R. If Ra is simple and $(Ra)^2 \neq 0$, then $Ra = Re$ for some $e \in \mathcal{I}(R)$. So $aR = gR$ for some $g \in \mathcal{I}(R)$. Thus f is also given by left multiplication by an element of R. If $(aR)^2 = 0$ or $(Ra)^2 = 0$, then $a \in J(R)$. Since R is right $J(R)$-IP-injective, then f is given by left multiplication by an element of R. The last statement is immediate. $\square$

The converse of Proposition 2.10 (2) is not true in general. The next example gives a right simple-injective ring which is not right $\mathcal{I}(R) \cup J(R)$-IP-injective.

Example 2.11 Let $R = \mathbf{Z} \propto \mathbf{Z}$ be the trivial extension of $\mathbf{Z}$ and $\mathbf{Z}$, i.e., $R = \mathbf{Z} \oplus \mathbf{Z}$ is an abelian group, with the usual addition and the following multiplication: $(r, x)(s, y) = (rs, ry + xs)$ for $r, x, s, y \in \mathbf{Z}$.

Since $\mathrm{Soc}(R_R) = 0, R$ is right simple injective. Let $y = 2 \propto 0 \in R$. Then $r(y) = 0$, but y is not a right unit. So R is not right $GC2$ by [2, Proposition 2.2], hence R is not right $\mathcal{I}(R) \cup J(R)$-IP-injective by Proposition 2.10 (1).

Corollary 2.12 *Let R be a right Kasch, right $\mathcal{I}(R) \cup J(R)$-IP-injective ring. Then*

(1) $r(l(I)) = I$ for every right ideal I of R;

(2) $J(R) = r(\mathrm{Soc}(R_R)) = Z({}_RR)$;

(3) R is right JP-injective in case R is semilocal.

Proof Since R is right simple-injective by Proposition 2.10 (2), then (1) and (2) follow from [7, Lemma 2.1].

(3) If R is semilocal, then R is left Kasch by [7, Lemma 2.2]. Thus R is right JP-injective by Proposition 2.9. □

Recall that a ring R is called semiregular if $R/J(R)$ is regular and idempotents of $R/J(R)$ can be lifted to idempotents of R. Following [8], a ring R is called right I-semiregular where I is an ideal of R if, for any $a \in R$, there exists $e^2 = e \in aR$ with $a - ea \in I$.

Proposition 2.13 *Assume that I is an ideal of a ring R, then every right I-semiregular right $\mathcal{I}(R) \cup I$-IP-injective ring R is right IP-injective. In particular, every semiregular right $\mathcal{I}(R) \cup J(R)$-IP-injective ring R is right IP-injective.*

Proof Let f be any right R-homomorphism from any right ideal of R into R_R with $\mathrm{Im}f = aR$, $a \in R$. Since R is right I-semiregular, we have $aR = eR \oplus bR$ with $e \in \mathcal{I}(R)$ and $b \in I$ by [8, Theorem 1.2]. Let $\pi_1 : aR \to eR$ and $\pi_2 : aR \to bR$ be canonical projections. Then $\mathrm{Im}\pi_1 f = eR$ and $\mathrm{Im}\pi_2 f = bR$. By hypothesis, $\pi_i f$ is given by left multiplication by element c_i of R with $c_i \in R$, $i = 1, 2$. It follows that $f = \pi_1 f + \pi_2 f$ is given by left multiplication by $c_1 + c_2$, so R is right IP-injective.

The last statement follows from the fact that R is semiregular if and only if R is $J(R)$-semiregular[8]. □

It is well-known that a ring R is a quasi-Frobenius ring if and only if R is a left Noetherian right self-injective ring. This result can be improved as follows.

Theorem 2.14 *A ring R is a quasi-Frobenius ring if and only if R is a left Noetherian, right ACS (i.e., for any $a \in R$, $r(a)$ is an essential submodule of a direct summand of R_R) right $\mathcal{I}(R) \cup J(R)$-IP-injective ring.*

Proof One direction is clear. Now, assume that R is a left Noetherian, right ACS right $\mathcal{I}(R) \cup J(R)$-IP-injective ring. By Proposition 2.10 (1), R is a right $C2$-ring. Since R is right ACS, then it is semiregular by [8, Theorem 2.4]. It follows that R is a right IP-injective ring by Proposition 2.13. Therefore R is a quasi-Frobenius ring by [4, Theorem 2.7]. □

References:

[1] NICHOLSON W K, YOUSIF M F. *Principally injective rings* [J]. J. Algebra, 1995, **174:** 77–93.

[2] YOUSIF M F, ZHOU Yi-qiang. *Rings for which certain elements have the principal extension property* [J]. Algebra Colloq., 2003, **10**: 501–512.

[3] NICHOLSON W K, YOUSIF M F. *On perfect simple-injective rings* [J]. Proc. Amer. Math. Soc., 1997, **125**:

979–985.

[4] CHEN Jian-long, DING Nan-qing, YOUSIF M F. *On a generalization of injective rings* [J]. Comm. Algebra, 2003, **31**: 5105–5116.

[5] ANDERSON F W, FULLER K R. *Rings and Categories of Modules* [M], Springer-Verlag: New York, USA, 1974.

[6] NICHOLSON W K, YOUSIF M F. *On quasi-Frobenius rings* [C]. International Symposium on Ring Theory (Kyongju, 1999), 245–277, Trends Math., Birkhäuser Boston, Boston, MA, 2001.

[7] CHEN Jian-long, DING Nan-qing, YOUSIF M F. *On generalizations of PF-rings* [J], Comm. Algebra, 2004, **32**: 521–533.

[8] NICHOLSON W K, YOUSIF M F. *Weakly continuous and $C2$-rings* [J], Comm. Algebra, 2001, **29**: 2429–2446.

广义 IP- 内射环

毛立新[1,2]，佟文廷[2]

(1. 南京工程学院基础部，江苏 南京 210013; 2. 南京大学数学系，江苏 南京 210093)

摘要: 对环 R, 令 $ip(R_R)=\{a\in R$: 任意一个从 R 的右理想到 R 且象为 aR 的模同态能开拓到 $R\}$. 众所周知， R 为右 IP- 内射环当且仅当 $R=ip(R_R)$, R 为右单 - 内射环当且仅当 $\{a\in R: aR \text{ is simple}\}\subseteq ip(R_R)$. 对环 R 的一个子集 S, 我们引进了 S-IP- 内射环的概念，即满足 $S\subseteq ip(R_R)$ 的环. 并得到了这种环的一些性质.

关键词: S-IP- 内射环; 单 - 内射环; $C2$- 环.